# Poking Holes in Schwarzschild-Tangherlini Black Holes

Based on arXiv:1907.07622 [gr-qc] (Journal of Mathematical Physics)

**Matthew S. Fox**
Harvey Mudd College, Claremont, CA
msfox@g.hmc.edu

**APS April Meeting**
21 April 2020

# Black Holes, $n = 3$

# Schwarzschild, Reissner, and Nordström

## Schwarzschild Black Hole


## Characteristic Properties

$$g_{\mathrm{S}}(M) \to -\left(1 - \frac{M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2^2$$

# Schwarzschild, Reissner, and Nordström

### Schwarzschild Black Hole


### Characteristic Properties

- electrically neutral

$$g_{\mathsf{S}}(M) \to -\left(1 - \frac{M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2^2$$

# Schwarzschild, Reissner, and Nordström

## Schwarzschild Black Hole


## Characteristic Properties

- electrically neutral
- static

$$g_{\mathrm{S}}(M) \to -\left(1-\tfrac{M}{r}\right)\mathrm{d}t^2 + \left(1-\tfrac{M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2$$

# Schwarzschild, Reissner, and Nordström

## Schwarzschild Black Hole


## Characteristic Properties

- electrically neutral
- static
- asymptotically flat

$$g_\mathsf{S}(M) \to -\left(1-\tfrac{M}{r}\right)\mathrm{d}t^2 + \left(1-\tfrac{M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2$$

# Schwarzschild, Reissner, and Nordström

## Reissner-Nordström Black Hole


## Characteristic Properties

- electrically charged

$$g_{\mathrm{RN}}(M, Q) \to -\left(1 - \frac{M}{r} + \frac{Q}{r^2}\right) \mathrm{d}t^2 + \left(1 - \frac{M}{r} + \frac{Q}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2^2$$

# Schwarzschild, Reissner, and Nordström

### Reissner-Nordström Black Hole


### Characteristic Properties

- electrically charged
- static

$$g_{\mathrm{RN}}(M, Q) \to -\left(1 - \frac{M}{r} + \frac{Q}{r^2}\right) \mathrm{d}t^2 + \left(1 - \frac{M}{r} + \frac{Q}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2^2$$

# Schwarzschild, Reissner, and Nordström

**Reissner-Nordström Black Hole**

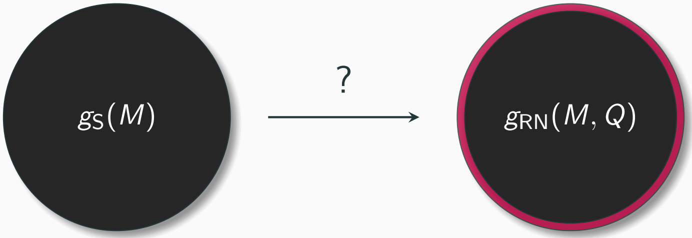

**Characteristic Properties**

- electrically charged
- static
- asymptotically flat

$$g_{\mathrm{RN}}(M, Q) \rightarrow -\left(1 - \frac{M}{r} + \frac{Q}{r^2}\right) \mathrm{d}t^2 + \left(1 - \frac{M}{r} + \frac{Q}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2^2$$

# The Problem

**Schwarzschild** **Reissner-Nordström**



# A Solution?

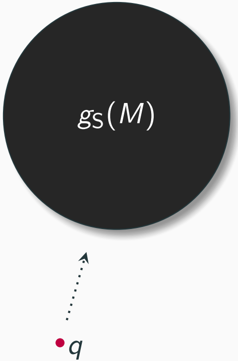
**Schwarzschild**


**Reissner-Nordström**


$q$

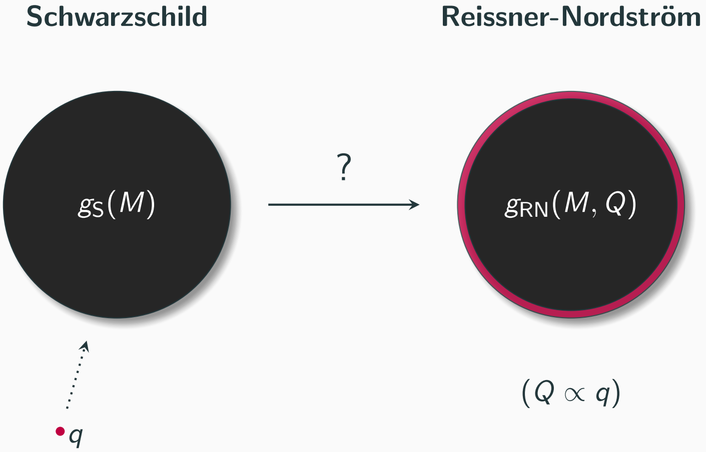
# A Solution?

**Schwarzschild**




**Reissner-Nordström**

# A Solution?



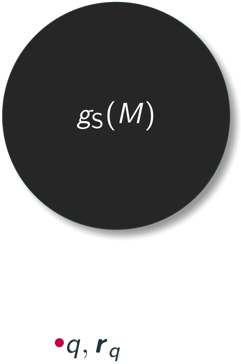
Schwarzschild
Reissner-Nordström
$g_{\mathrm{S}}(M)$
?
$g_{\mathrm{RN}}(M, Q)$
$q$
$(Q \propto q)$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 12, NUMBER 9 SEPTEMBER 1971

# Point Charge in the Vicinity of a Schwarzschild Black Hole*

Jeffrey M. Cohen
*Institute for Advanced Study, Princeton, New Jersey 08540*
and
Robert M. Wald†
*Joseph Henry Physical Laboratory, Princeton University, Princeton, New Jersey 08540*
(Received 1 February 1971)

**The electrostatic field of a point charge at rest in Schwarzschild space is derived. The solution is used to study the problem of a point charge slowly lowered into a nonrotating black hole. We find that the electric field of the charge remains well behaved as the charge is lowered and that all the multipole moments except the monopole fade away. We conclude that a Reissner–Nordstrom black hole is produced.**



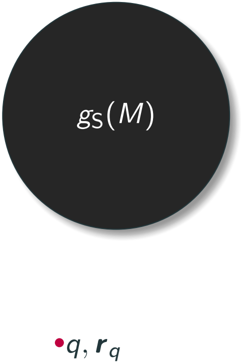
JOURNAL OF MATHEMATICAL PHYSICS VOLUME 12, NUMBER 9 SEPTEMBER 1971

# Point Charge in the Vicinity of a Schwarzschild Black Hole*

Jeffrey M. Cohen
*Institute for Advanced Study, Princeton, New Jersey 08540*
and
Robert M. Wald†
*Joseph Henry Physical Laboratory, Princeton University, Princeton, New Jersey 08540*

(Received 1 February 1971)

**The electrostatic field of a point charge at rest in Schwarzschild space is derived. The solution is used to study the problem of a point charge slowly lowered into a nonrotating black hole. We find that the electric field of the charge remains well behaved as the charge is lowered and that all the multipole moments except the monopole fade away. We conclude that a Reissner–Nordstrom black hole is produced.**




$$\Phi(\boldsymbol{r}, \boldsymbol{r}_q) \propto \frac{q}{r} + (r_q - r_s) \times \underbrace{\mathcal{O}(r^{-2})}_{\text{multipoles}}$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 12, NUMBER 9 SEPTEMBER 1971

# Point Charge in the Vicinity of a Schwarzschild Black Hole*

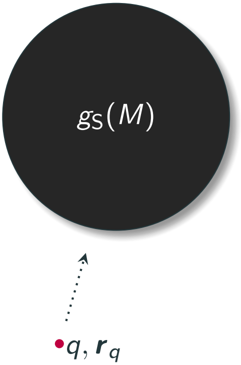
Jeffrey M. Cohen
*Institute for Advanced Study, Princeton, New Jersey 08540*

and

Robert M. Wald†
*Joseph Henry Physical Laboratory, Princeton University, Princeton, New Jersey 08540*

(Received 1 February 1971)

The electrostatic field of a point charge at rest in Schwarzschild space is derived. The solution is used to study the problem of a point charge slowly lowered into a nonrotating black hole. We find that the electric field of the charge remains well behaved as the charge is lowered and that all the multipole moments except the monopole fade away. We conclude that a Reissner–Nordstrom black hole is produced.




$$\Phi(\boldsymbol{r}, \boldsymbol{r}_q) \propto \frac{q}{r} + (r_q - r_s) \times \underbrace{\mathcal{O}(r^{-2})}_{\text{multipoles}}$$

$$\lim_{r_q \to r_s^+} \Phi(\boldsymbol{r}, \boldsymbol{r}_q) \propto \frac{q}{r}$$

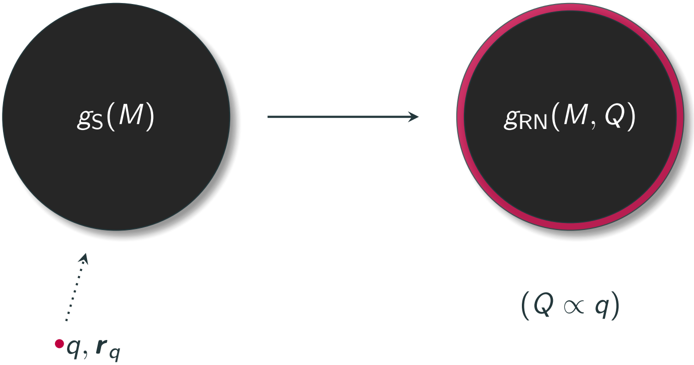
(spherical symmetry!)

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 12, NUMBER 9 SEPTEMBER 1971

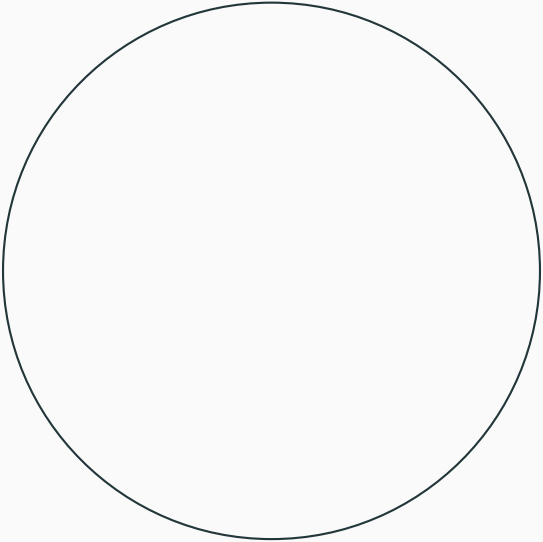
# Point Charge in the Vicinity of a Schwarzschild Black Hole*

Jeffrey M. Cohen
*Institute for Advanced Study, Princeton, New Jersey 08540*
and
Robert M. Wald†
*Joseph Henry Physical Laboratory, Princeton University, Princeton, New Jersey 08540*
(Received 1 February 1971)

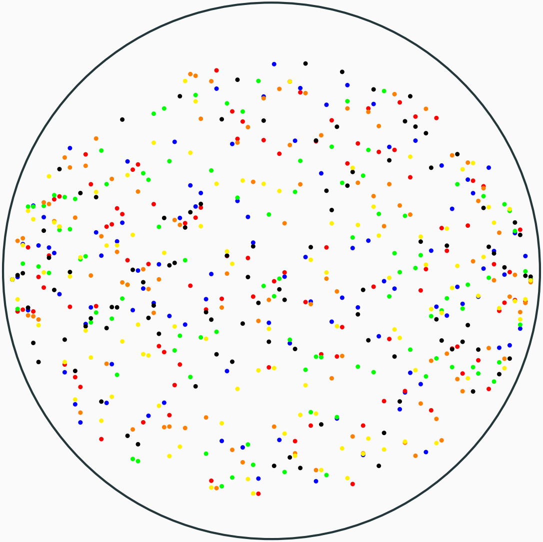
The electrostatic field of a point charge at rest in Schwarzschild space is derived. The solution is used to study the problem of a point charge slowly lowered into a nonrotating black hole. We find that the electric field of the charge remains well behaved as the charge is lowered and that all the multipole moments except the monopole fade away. We conclude that a Reissner–Nordstrom black hole is produced.



# The No-Hair Theorem



# The No-Hair Theorem



# The No-Hair Theorem

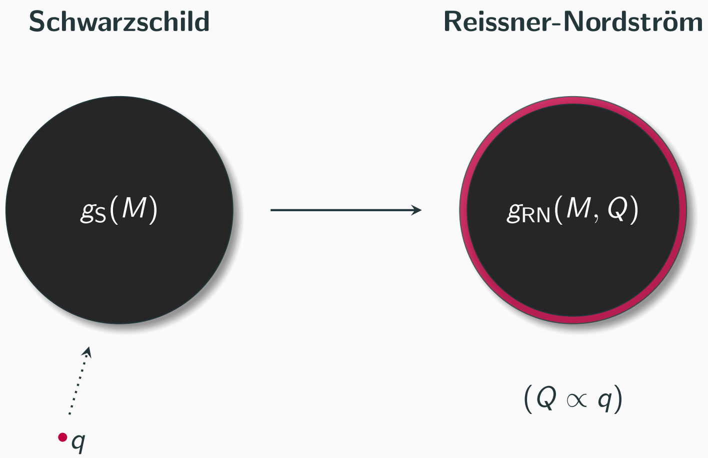

$$g = g(M, Q, J)\ \textbf{not}\ g(\text{red}, \text{blue}, \text{green}, \ldots)$$

W. Israel, *Phys. Rev.* **164**, 1776 (1967)

B. Carter, *Phys. Rev. Lett.* **26**, 331 (1971)

D. C. Robinson, *Phys. Rev. Lett.* **34**, 905 (1975)

# The "Sledgehammer" Solution



# Black Hole Topology, n $= 3$

**Theorem** (Hawking, 1972). Black hole boundary $\mathcal{B}$ satisfies

$$\int_{\mathcal{B}} R \, \mathrm{d}A > 0$$

S. W. Hawking, *Commun. Math. Phys.* **25**, 152 (1972)

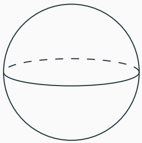
# Black Hole Topology, $n = 3$

**Theorem** (Hawking, 1972). Black hole boundary $\mathcal{B}$ satisfies

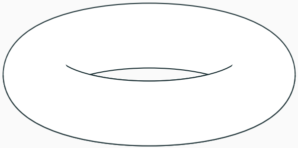
$$\int_{\mathcal{B}} R \, \mathrm{d}A > 0$$

**Theorem** (Gauss-Bonnet). A closed 2D Riemannian manifold $\mathcal{B}$ with scalar curvature $R$ satisfies

$$\int_{\mathcal{B}} R \, \mathrm{d}A = \pi(2 - 2 \times \text{genus})$$



genus 0



genus 1

. . .

S. W. Hawking, *Commun. Math. Phys.* **25**, 152 (1972)

# Black Holes, $n > 3$

# Black Hole Topology, n $> 3$

Let $g$ be a Riemannian metric on $\mathcal{B}$ and define **conformal class**

$$[g] \equiv \left\{ h = e^{2f} g \,\middle|\, f : \mathcal{B} \to \mathbf{R} \right\}$$

J. M. Lee and T. H. Parker, *Bull. Am. Math. Soc* **17**, 37 (1987)

# Black Hole Topology, n $> 3$

Let $g$ be a Riemannian metric on $\mathcal{B}$ and define **conformal class**

$$[g] \equiv \left\{ h = e^{2f} g \,\middle|\, f : \mathcal{B} \to \mathbf{R} \right\}$$

Associate to each conformal class $[g]$ on $\mathcal{B}$ a **Yamabe invariant**

$$\lambda([g]) = \inf_{h \in [g]} \left( \int_{\mathcal{B}} R_h \, \mathrm{d}V_h \right) \left( \int_{\mathcal{B}} \mathrm{d}V_h \right)^{-\frac{n-3}{n-1}}$$

---

J. M. Lee and T. H. Parker, *Bull. Am. Math. Soc* **17**, 37 (1987)

# Black Hole Topology, n $> 3$

Let $g$ be a Riemannian metric on $\mathcal{B}$ and define **conformal class**

$$[g] \equiv \left\{ h = e^{2f} g \,\middle|\, f : \mathcal{B} \to \mathbf{R} \right\}$$

Associate to each conformal class $[g]$ on $\mathcal{B}$ a **Yamabe invariant**

$$\lambda([g]) = \inf_{h \in [g]} \left( \int_{\mathcal{B}} R_h \, \mathrm{d} V_h \right) \left( \int_{\mathcal{B}} \mathrm{d} V_h \right)^{-\frac{n-3}{n-1}}$$

The maximal Yamabe invariant of $\mathcal{B}$ is then

$$\sigma(\mathcal{B}) = \sup_{[g]} \lambda([g])$$

J. M. Lee and T. H. Parker, *Bull. Am. Math. Soc* **17**, 37 (1987)

# Black Hole Topology, n $> 3$

Let $g$ be a Riemannian metric on $\mathcal{B}$ and define **conformal class**

$$[g] \equiv \left\{ h = e^{2f} g \,\middle|\, f : \mathcal{B} \to \mathbf{R} \right\}$$

Associate to each conformal class $[g]$ on $\mathcal{B}$ a **Yamabe invariant**

$$\lambda([g]) = \inf_{h \in [g]} \left( \int_{\mathcal{B}} R_h \, \mathrm{d}V_h \right) \left( \int_{\mathcal{B}} \mathrm{d}V_h \right)^{-\frac{n-3}{n-1}}$$

The maximal Yamabe invariant of $\mathcal{B}$ is then

$$\sigma(\mathcal{B}) = \sup_{[g]} \lambda([g])$$

**Theorem** (Lee and Parker, 1987). For compact, Riemannian $\mathcal{B}$, $\sigma(\mathcal{B}) > 0$ if and only if $\mathcal{B}$ carries metric of positive scalar curvature

---

J. M. Lee and T. H. Parker, *Bull. Am. Math. Soc* **17**, 37 (1987)

# Black Hole Topology, n $> 3$

**Theorem** (Galloway *et al.*, 2006). Black hole boundary $\mathcal{B}$ satisfies

$$\sigma(\mathcal{B}) > 0$$

i.e. $\mathcal{B}$ carries metric of positive scalar curvature

---

G. J. Galloway and R. Schoen, *Commun. Math. Phys.* **266**, 571 (2006)

R. Emparan and H. S. Reall, *Phys. Rev. Lett.* **88**, 101101 (2002)

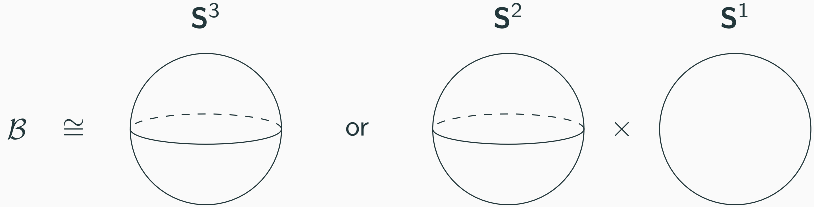
# Black Hole Topology, n $> 3$

**Theorem** (Galloway *et al.*, 2006). Black hole boundary $\mathcal{B}$ satisfies

$$\sigma(\mathcal{B}) > 0$$

i.e. $\mathcal{B}$ carries metric of positive scalar curvature

**Example** (Emparan and Raell, 2002). If $n = 4$,

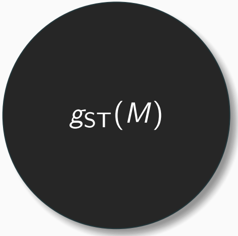



G. J. Galloway and R. Schoen, *Commun. Math. Phys.* **266**, 571 (2006)

R. Emparan and H. S. Raell, *Phys. Rev. Lett.* **88**, 101101 (2002)

# Uniqueness Theorems, $\mathcal{B} \cong \mathbf{S}^{n-1}$

## Schwarzschild-Tangherlini (ST)




## Characteristic Properties

- electrically neutral
- static
- asymptotically flat

$$g_{\mathrm{ST}}(M) \to -\left(1 - \frac{M}{r^{n-2}}\right) \mathrm{d}t^2 + \left(1 - \frac{M}{r^{n-2}}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{n-1}^2$$

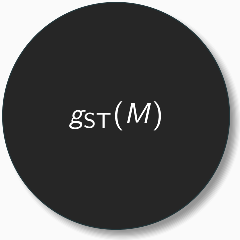
---

S. Hwang, *Geometriae Dedicata* **71**, 5 (1998)

M. Rogatko, *Phys. Rev. D* **67**, 084025 (2003); *Phys. Rev. D* **73**, 124027 (2006)

# Uniqueness Theorems, $\mathcal{B} \cong \mathbf{S}^{n-1}$

## Schwarzschild-Tangherlini (ST)




## Characteristic Properties

- electrically neutral
- static
- asymptotically flat
- non-degenerate horizon

$$g_{\mathrm{ST}}(M) \to -\left(1 - \frac{M}{r^{n-2}}\right) \mathrm{d}t^2 + \left(1 - \frac{M}{r^{n-2}}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{n-1}^2$$

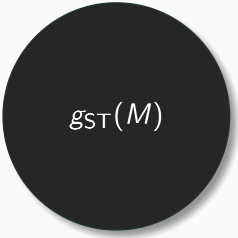
---

S. Hwang, *Geometriae Dedicata* **71**, 5 (1998)

M. Rogatko, *Phys. Rev. D* **67**, 084025 (2003); *Phys. Rev. D* **73**, 124027 (2006)

# Uniqueness Theorems, $\mathcal{B} \cong \mathbf{S}^{n-1}$

## Schwarzschild-Tangherlini (ST)




## Characteristic Properties

- electrically neutral
- static
- asymptotically flat
- non-degenerate horizon
- $\mathcal{B} \cong \mathbf{S}^{n-1}$

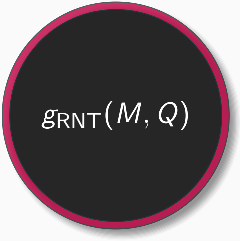
$$g_{\mathrm{ST}}(M) \to -\left(1 - \frac{M}{r^{n-2}}\right) \mathrm{d}t^2 + \left(1 - \frac{M}{r^{n-2}}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{n-1}$$

S. Hwang, *Geometriae Dedicata* **71**, 5 (1998)

M. Rogatko, *Phys. Rev. D* **67**, 084025 (2003); *Phys. Rev. D* **73**, 124027 (2006)

# Uniqueness Theorems, $\mathcal{B} \cong \mathbf{S}^{n-1}$

## Reissner-Nordström-Tangherlini (RNT)




## Characteristic Properties

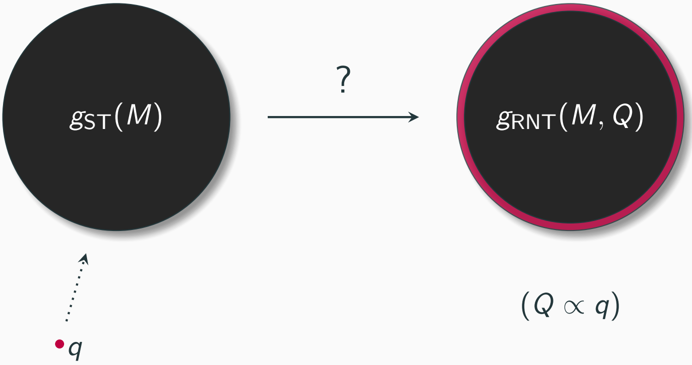
- electrically charged
- static
- asymptotically flat
- non-degenerate horizon
- $\mathcal{B} \cong \mathbf{S}^{n-1}$

$$g_{\mathrm{RNT}}(M, Q) \to -\left(1 - \frac{M}{r^{n-2}} + \frac{Q}{r^{2n-4}}\right) \mathrm{d}t^2 + \left(1 - \frac{M}{r^{n-2}} + \frac{Q}{r^{2n-4}}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{n-1}$$

G. W. Gibbons, D. Ida, and T. Shiromizu, *Phys. Rev. D* **66**, 044010 (2002)

# Same Problem, n > 3

**Schwarzschild-Tangherlini** **Reissner-Nordström-Tangherlini**



# Electrostatics in ST Geometry

**Task**

Solve Maxwell's equations in ST spacetime $(\mathcal{M}^{n+1}, g_{\mathsf{ST}})$,

$$j^\nu = \frac{1}{\sqrt{|\det g_{\mathsf{ST}}|}} \partial_\mu \left( \sqrt{|\det g_{\mathsf{ST}}|} F^{\mu\nu} \right)$$

# Electrostatics in ST Geometry

**Task**

Solve Maxwell's equations in ST spacetime $(\mathcal{M}^{n+1}, g_{\mathsf{ST}})$,

$$j^\nu = \frac{1}{\sqrt{|\det g_{\mathsf{ST}}|}} \partial_\mu \left( \sqrt{|\det g_{\mathsf{ST}}|} F^{\mu\nu} \right)$$

**Approach**

Assuming electrostatic conditions with charge $q$ at position $\boldsymbol{r}_q$,

$$\Delta_{\mathcal{M}^{n+1}} \Phi(\boldsymbol{r}, \boldsymbol{r}_q) = q g_{00} \delta(\boldsymbol{r} - \boldsymbol{r}_q) + \frac{r_s^{n-2}(n-2)}{r^{n-1}} \partial_r \Phi(\boldsymbol{r}, \boldsymbol{r}_q)$$

Poisson's equation!

# Electrostatics in ST Geometry

## Task

Solve Maxwell's equations in ST spacetime $(\mathcal{M}^{n+1}, g_{\mathsf{ST}})$,

$$j^\nu = \frac{1}{\sqrt{|\det g_{\mathsf{ST}}|}} \partial_\mu \left( \sqrt{|\det g_{\mathsf{ST}}|} F^{\mu\nu} \right)$$

## Approach

Assuming electrostatic conditions with charge $q$ at position $\boldsymbol{r}_q$,

$$\Delta_{\mathcal{M}^{n+1}} \Phi(\boldsymbol{r}, \boldsymbol{r}_q) = q g_{00} \delta(\boldsymbol{r} - \boldsymbol{r}_q) + \frac{r_s^{n-2}(n-2)}{r^{n-1}} \partial_r \Phi(\boldsymbol{r}, \boldsymbol{r}_q)$$

Poisson's equation! Use multipole ansatz

$$\Phi(\boldsymbol{r}, \boldsymbol{r}_q) = \sum_{k \geq 0} R_k(r, r_q) Y_k(\varphi)$$

# Electrostatics in ST Geometry

## Radial Equation

Upshot,

$$\lim_{r_q \to r_s^+} R_k(r, r_q) \propto \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \in \mathbf{Z}^+ \text{ and } (n-2) \mid k \\ \frac{\Gamma\left(-\frac{2k}{n-2}\right)}{\Gamma\left(-\frac{k}{n-2}\right)\Gamma\left(1-\frac{k}{n-2}\right)} & \text{otherwise} \end{cases}$$

M. S. Fox, *J. Math. Phys.* **60**, 102502 (2019)

## Radial Equation

Upshot,

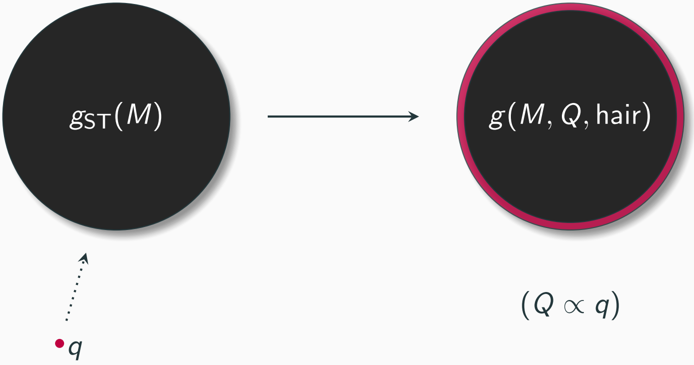
$$\lim_{r_q \to r_s^+} R_k(r, r_q) \propto \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \in \mathbf{Z}^+ \text{ and } (n-2) \mid k \\ \dfrac{\Gamma\left(-\frac{2k}{n-2}\right)}{\Gamma\left(-\frac{k}{n-2}\right)\Gamma\left(1-\frac{k}{n-2}\right)} & \text{otherwise} \end{cases}$$

There exist multipole moments ("multipole hair") when $n > 3$

M. S. Fox, *J. Math. Phys.* **60**, 102502 (2019)

# Multipole Hair of ST Black Holes

**Schwarzschild-Tangherlini** | **Something Else** ~~**Reissner-Nordström-Tangherlini**~~




M. S. Fox, *J. Math. Phys.* **60**, 102502 (2019)

# Charge-Induced Topology Change

## ST + Charge Black Hole


## Characteristic Properties

- electrically charged

M. S. Fox, *J. Math. Phys.* **60**, 102502 (2019)

[†] M. Rogatko, *Phys. Rev. D* **67**, 084025 (2003); *Phys. Rev. D* **73**, 124027 (2006)

# Charge-Induced Topology Change

## ST + Charge Black Hole


## Characteristic Properties

- electrically charged
- static

M. S. Fox, *J. Math. Phys.* **60**, 102502 (2019)

† M. Rogatko, *Phys. Rev. D* **67**, 084025 (2003); *Phys. Rev. D* **73**, 124027 (2006)

# Charge-Induced Topology Change

## ST + Charge Black Hole


## Characteristic Properties

- electrically charged
- static
- asymptotically flat

M. S. Fox, *J. Math. Phys.* **60**, 102502 (2019)

[†] M. Rogatko, *Phys. Rev. D* **67**, 084025 (2003); *Phys. Rev. D* **73**, 124027 (2006)

# Charge-Induced Topology Change

## ST + Charge Black Hole


## Characteristic Properties

- electrically charged
- static
- asymptotically flat
- non-degenerate horizon (likely†)

---

M. S. Fox, *J. Math. Phys.* **60**, 102502 (2019)

† M. Rogatko, *Phys. Rev. D* **67**, 084025 (2003); *Phys. Rev. D* **73**, 124027 (2006)

# Charge-Induced Topology Change

## ST + Charge Black Hole


## Characteristic Properties

- electrically charged
- static
- asymptotically flat
- non-degenerate horizon (likely†)
- $\mathcal{B} \ncong \mathbf{S}^{n-1}$

Final state **not** topologically spherical

M. S. Fox, *J. Math. Phys.* **60**, 102502 (2019)

† M. Rogatko, *Phys. Rev. D* **67**, 084025 (2003); *Phys. Rev. D* **73**, 124027 (2006)

# Acknowledgements


- Jason Gallicchio (HMC)
- Brian Shuve (HMC)
- Thomas Moore (Pomona College)

# Radial Equation

$$R_k'' + \frac{n-1}{r} R_k' - \frac{k(k+n-2)r^{n-4}}{r^{n-2} - r_s^{n-2}} R_k = 0$$

Define $\rho = \left(\frac{r_s}{r}\right)^{n-2}$,

$$\rho^2(\rho - 1)\ddot{R}_k + \frac{k(k+n-2)}{(n-2)^2} R_k = 0$$

Hypergeometric function! Solve around $\rho = 0$ so $r \in (r_s, +\infty)$:

$$R_k^{(\alpha)}(r, r_s) = \sum_{m \geq 0} \alpha_{k,m} \rho^{m + \chi_k^+}$$

$$R_k^{(\bar{\alpha})}(r, r_s) = \sum_{m \geq 0} \bar{\alpha}_{k,m} \rho^{m - \chi_k^-}$$

# Radial Equation

$$R_k^{(\alpha)}(r, r_s) = \sum_{m\geq 0} \alpha_{k,m}\rho^{m+\chi_k^+}$$

$$R_k^{(\bar{\alpha})}(r, r_s) = \sum_{m\geq 0} \bar{\alpha}_{k,m}\rho^{m-\chi_k^-}$$

Here, $\chi_k^+$ and $\chi_k^-$ are roots of indicial equation for radial equation,

$$\chi_k^{\pm} = \frac{1\pm 1}{2} + \frac{k}{n-2}$$

Then,

$$\tilde{\alpha}_{k,m} \equiv \frac{\alpha_{k,m}}{\alpha_{k,0}} = \frac{(\chi_k^+)_m(\chi_k^-)_m}{m!(2\chi_k^+)_m}$$

and

$$\tilde{\bar{\alpha}}_{k,m} \equiv \frac{\bar{\alpha}_{k,m}}{\bar{\alpha}_{k,0}} = \frac{(-\chi_k^-)_m(-\chi_k^+)_m}{m!(-2\chi_k^-)_m}$$

# Radial Equation

Matching asymptotic part with electrostatics in $n+1$ Minkowski,

$$R_k^{(\alpha)}(r, r_s) = r_s^{-(k+n-2)} \sum_{m \geq 0} \tilde{\alpha}_{k,m} \left(\frac{r_s}{r}\right)^{k+(m+1)(n-2)}$$

$$R_k^{(\bar{\alpha})}(r, r_s) = r_s^{k} \sum_{m \geq 0} \tilde{\bar{\alpha}}_{k,m} \left(\frac{r}{r_s}\right)^{k-m(n-2)}$$

# Green's Function

$$\Phi(\boldsymbol{r}, \boldsymbol{r}_q) \propto q \sum_{k \in \Lambda} \sum_{l=1}^{\Gamma_k} \sum_{m \geq 0} \frac{\tilde{\alpha}_{k,m} R_k^{(\bar{\alpha})} Y_k^{*l}(\varphi_q) Y_k^l(\varphi)}{r_s^{k+n-2}(2k+n-2)} \left(\frac{r_s}{r}\right)^{k+(m+1)(n-2)}$$

Here:

$$\Lambda = \{k \in \mathbf{Z}^+ : (n-2) \nmid k\} \cup \{0\}$$

$$\Gamma_k = \frac{(2k+n-2)(n+k-3)!}{k!(n-2)!}$$

There exists a $\mu_k$-pole moment for each $k \in \Lambda$ provided

$$\mu_k \equiv k \pmod{n-2}$$