

Electric Multipole Fields Near Schwarzschild-Tangherlini Black Holes

1907.07622 (*J. Math. Phys.*); 2003.12179 (*Phys. Rev. D*)

Matthew S. Fox

Harvey Mudd College, Claremont, CA

msfox@hmc.edu

Midwest Relativity Meeting

23 October 2020



Black Holes, $n = 3$

Schwarzschild, Reissner, and Nordström

Schwarzschild Black Hole



Characteristic Properties

$$g_S(M) \rightarrow -\left(1 - \frac{M}{r}\right) dt^2 + \left(1 - \frac{M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

Schwarzschild, Reissner, and Nordström

Schwarzschild Black Hole



Characteristic Properties

- electrically neutral

$$g_S(M) \rightarrow -\left(1 - \frac{M}{r}\right) dt^2 + \left(1 - \frac{M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

Schwarzschild, Reissner, and Nordström

Schwarzschild Black Hole



Characteristic Properties

- electrically neutral
- static

$$g_S(M) \rightarrow -\left(1 - \frac{M}{r}\right) dt^2 + \left(1 - \frac{M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

Schwarzschild, Reissner, and Nordström

Schwarzschild Black Hole



Characteristic Properties

- electrically neutral
- static
- asymptotically flat

$$g_S(M) \rightarrow -\left(1 - \frac{M}{r}\right) dt^2 + \left(1 - \frac{M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

Schwarzschild, Reissner, and Nordström

Reissner-Nordström Black Hole



Characteristic Properties

- electrically charged

$$g_{\text{RN}}(M, Q) \rightarrow -\left(1 - \frac{M}{r} + \frac{Q}{r^2}\right) dt^2 + \left(1 - \frac{M}{r} + \frac{Q}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

Reissner-Nordström Black Hole



Characteristic Properties

- electrically charged
- static

$$g_{\text{RN}}(M, Q) \rightarrow -\left(1 - \frac{M}{r} + \frac{Q}{r^2}\right) dt^2 + \left(1 - \frac{M}{r} + \frac{Q}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

Schwarzschild, Reissner, and Nordström

Reissner-Nordström Black Hole



Characteristic Properties

- electrically charged
- static
- asymptotically flat

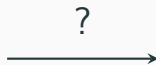
$$g_{\text{RN}}(M, Q) \rightarrow -\left(1 - \frac{M}{r} + \frac{Q}{r^2}\right) dt^2 + \left(1 - \frac{M}{r} + \frac{Q}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

The Problem

Schwarzschild



Reissner-Nordström



A Solution?

Schwarzschild



Reissner-Nordström



• q

A Solution?

Schwarzschild

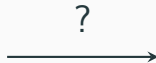


Reissner-Nordström



A Solution?

Schwarzschild



Reissner-Nordström



$$(Q \propto q)$$

Point Charge in the Vicinity of a Schwarzschild Black Hole*

Jeffrey M. Cohen

Institute for Advanced Study, Princeton, New Jersey 08540

and

Robert M. Wald[†]

Joseph Henry Physical Laboratory, Princeton University, Princeton, New Jersey 08540

(Received 1 February 1971)

The electrostatic field of a point charge at rest in Schwarzschild space is derived. The solution is used to study the problem of a point charge slowly lowered into a nonrotating black hole. We find that the electric field of the charge remains well behaved as the charge is lowered and that all the multipole moments except the monopole fade away. We conclude that a Reissner-Nordstrom black hole is produced.



• q, r_q

Point Charge in the Vicinity of a Schwarzschild Black Hole*

Jeffrey M. Cohen

Institute for Advanced Study, Princeton, New Jersey 08540

and

Robert M. Wald[†]

Joseph Henry Physical Laboratory, Princeton University, Princeton, New Jersey 08540

(Received 1 February 1971)

The electrostatic field of a point charge at rest in Schwarzschild space is derived. The solution is used to study the problem of a point charge slowly lowered into a nonrotating black hole. We find that the electric field of the charge remains well behaved as the charge is lowered and that all the multipole moments except the monopole fade away. We conclude that a Reissner-Nordstrom black hole is produced.



$$\Phi(r, r_q) \propto \frac{q}{r} + (r_q - r_s) \times \underbrace{\mathcal{O}(r^{-2})}_{\text{multipoles}}$$

• q, r_q

Point Charge in the Vicinity of a Schwarzschild Black Hole*

Jeffrey M. Cohen

Institute for Advanced Study, Princeton, New Jersey 08540

and

Robert M. Wald[†]

Joseph Henry Physical Laboratory, Princeton University, Princeton, New Jersey 08540

(Received 1 February 1971)

The electrostatic field of a point charge at rest in Schwarzschild space is derived. The solution is used to study the problem of a point charge slowly lowered into a nonrotating black hole. We find that the electric field of the charge remains well behaved as the charge is lowered and that all the multipole moments except the monopole fade away. We conclude that a Reissner-Nordstrom black hole is produced.



$$\Phi(\mathbf{r}, \mathbf{r}_q) \propto \frac{q}{r} + (\mathbf{r}_q - \mathbf{r}_s) \times \underbrace{\mathcal{O}(r^{-2})}_{\text{multipoles}}$$

$$\lim_{r_q \rightarrow r_s^+} \Phi(\mathbf{r}, \mathbf{r}_q) \propto \frac{q}{r}$$

(spherical symmetry!)

Point Charge in the Vicinity of a Schwarzschild Black Hole*

Jeffrey M. Cohen

Institute for Advanced Study, Princeton, New Jersey 08540

and

Robert M. Wald[†]

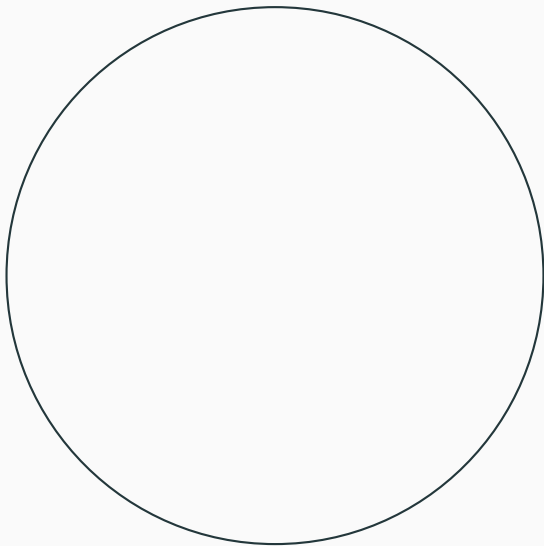
Joseph Henry Physical Laboratory, Princeton University, Princeton, New Jersey 08540

(Received 1 February 1971)

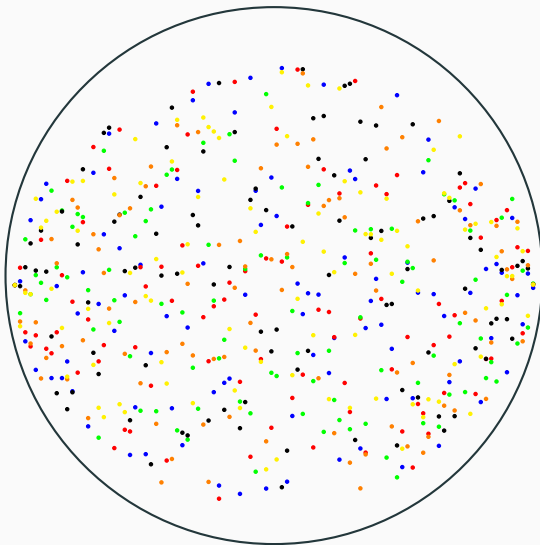
The electrostatic field of a point charge at rest in Schwarzschild space is derived. The solution is used to study the problem of a point charge slowly lowered into a nonrotating black hole. We find that the electric field of the charge remains well behaved as the charge is lowered and that all the multipole moments except the monopole fade away. We conclude that a Reissner-Nordstrom black hole is produced.



The No-Hair Theorem



The No-Hair Theorem



The No-Hair Theorem



$$g = g(M, Q, J) \text{ not } g(\text{red}, \text{blue}, \text{green}, \dots)$$

W. Israel, *Phys. Rev.* **164**, 1776 (1967)

B. Carter, *Phys. Rev. Lett.* **26**, 331 (1971)

D. C. Robinson, *Phys. Rev. Lett.* **34**, 905 (1975)

The “Sledgehammer” Solution

Schwarzschild



Reissner-Nordström



$$(Q \propto q)$$

Black Hole Topology, $n = 3$

Theorem (Hawking, 1972). Black hole boundary $\mathcal{B}$ satisfies

$$\int_{\mathcal{B}} R \, dA > 0$$

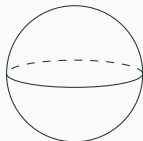
Black Hole Topology, $n = 3$

Theorem (Hawking, 1972). Black hole boundary $\mathcal{B}$ satisfies

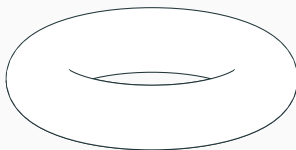
$$\int_{\mathcal{B}} R \, dA > 0$$

Theorem (Gauss-Bonnet). A closed 2D Riemannian manifold $\mathcal{B}$ with scalar curvature R satisfies

$$\int_{\mathcal{B}} R \, dA = \pi(2 - 2 \times \text{genus})$$



genus 0



genus 1

...

Black Holes, $n > 3$

Black Hole Topology, $n > 3$

Let g be a Riemannian metric on $\mathcal{B}$ and define **conformal class**

$$[g] \equiv \left\{ h = e^{2f} g \mid f : \mathcal{B} \rightarrow \mathbf{R} \right\}$$

Black Hole Topology, $n > 3$

Let g be a Riemannian metric on $\mathcal{B}$ and define **conformal class**

$$[g] \equiv \left\{ h = e^{2f} g \mid f : \mathcal{B} \rightarrow \mathbf{R} \right\}$$

Associate to each conformal class $[g]$ on $\mathcal{B}$ a **Yamabe invariant**

$$\lambda([g]) = \inf_{h \in [g]} \left(\int_{\mathcal{B}} R_h dV_h \right) \left(\int_{\mathcal{B}} dV_h \right)^{-\frac{n-3}{n-1}}$$

Black Hole Topology, $n > 3$

Let g be a Riemannian metric on $\mathcal{B}$ and define **conformal class**

$$[g] \equiv \left\{ h = e^{2f} g \mid f : \mathcal{B} \rightarrow \mathbf{R} \right\}$$

Associate to each conformal class $[g]$ on $\mathcal{B}$ a **Yamabe invariant**

$$\lambda([g]) = \inf_{h \in [g]} \left(\int_{\mathcal{B}} R_h dV_h \right) \left(\int_{\mathcal{B}} dV_h \right)^{-\frac{n-3}{n-1}}$$

The maximal Yamabe invariant of $\mathcal{B}$ is then

$$\sigma(\mathcal{B}) = \sup_{[g]} \lambda([g])$$

Black Hole Topology, $n > 3$

Let g be a Riemannian metric on $\mathcal{B}$ and define **conformal class**

$$[g] \equiv \left\{ h = e^{2f} g \mid f : \mathcal{B} \rightarrow \mathbf{R} \right\}$$

Associate to each conformal class $[g]$ on $\mathcal{B}$ a **Yamabe invariant**

$$\lambda([g]) = \inf_{h \in [g]} \left(\int_{\mathcal{B}} R_h dV_h \right) \left(\int_{\mathcal{B}} dV_h \right)^{-\frac{n-3}{n-1}}$$

The maximal Yamabe invariant of $\mathcal{B}$ is then

$$\sigma(\mathcal{B}) = \sup_{[g]} \lambda([g])$$

Theorem (Lee and Parker, 1987). For compact, Riemannian $\mathcal{B}$, $\sigma(\mathcal{B}) > 0$ if and only if $\mathcal{B}$ carries metric of positive scalar curvature

Black Hole Topology, $n > 3$

Theorem (Galloway *et al.*, 2006). Black hole boundary $\mathcal{B}$ satisfies

$$\sigma(\mathcal{B}) > 0$$

i.e. $\mathcal{B}$ carries metric of positive scalar curvature

G. J. Galloway and R. Schoen, *Commun. Math. Phys.* **266**, 571 (2006)

R. Emparan and H. S. Raell, *Phys. Rev. Lett.* **88**, 101101 (2002)

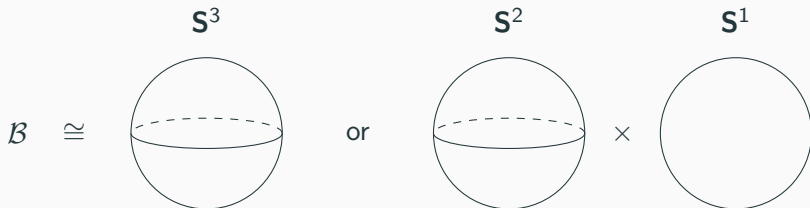
Black Hole Topology, $n > 3$

Theorem (Galloway *et al.*, 2006). Black hole boundary $\mathcal{B}$ satisfies

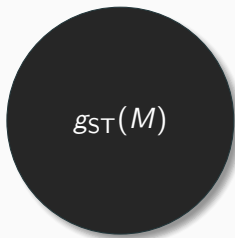
$$\sigma(\mathcal{B}) > 0$$

i.e. $\mathcal{B}$ carries metric of positive scalar curvature

Example (Emparan and Raell, 2002). If $n = 4$,



Schwarzschild-Tangherlini (ST)

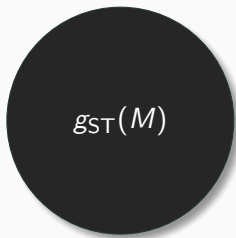


Characteristic Properties

- electrically neutral
- static
- asymptotically flat

$$g_{\text{ST}}(M) \rightarrow - \left(1 - \frac{M}{r^{n-2}}\right) dt^2 + \left(1 - \frac{M}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}^2$$

Schwarzschild-Tangherlini (ST)

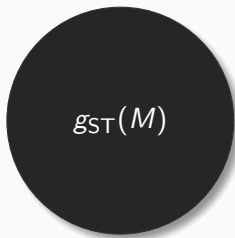


Characteristic Properties

- electrically neutral
- static
- asymptotically flat
- non-degenerate horizon

$$g_{\text{ST}}(M) \rightarrow - \left(1 - \frac{M}{r^{n-2}}\right) dt^2 + \left(1 - \frac{M}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}^2$$

Schwarzschild-Tangherlini (ST)

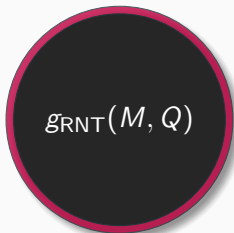


Characteristic Properties

- electrically neutral
- static
- asymptotically flat
- non-degenerate horizon
- $\mathcal{B} \cong \mathbf{S}^{n-1}$

$$g_{\text{ST}}(M) \rightarrow -\left(1 - \frac{M}{r^{n-2}}\right) dt^2 + \left(1 - \frac{M}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}^2$$

Reissner-Nordström-Tangherlini (RNT)



Characteristic Properties

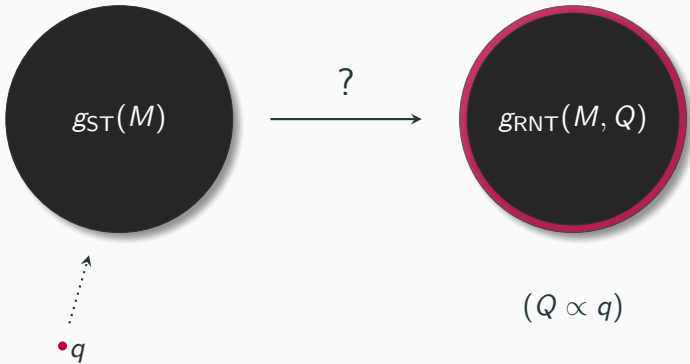
- electrically charged
- static
- asymptotically flat
- non-degenerate horizon
- $\mathcal{B} \cong \mathbf{S}^{n-1}$

$$g_{\text{RNT}}(M, Q) \rightarrow - \left(1 - \frac{M}{r^{n-2}} + \frac{Q}{r^{2n-4}}\right) dt^2 + \left(1 - \frac{M}{r^{n-2}} + \frac{Q}{r^{2n-4}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}^2$$

Same Problem, $n > 3$

Schwarzschild-Tangherlini

Reissner-Nordström-Tangherlini



Electrostatics in ST Geometry

Task

Solve Maxwell's equations in ST spacetime $(\mathcal{M}^{n+1}, g_{\text{ST}})$,

$$j^\nu = \frac{1}{\sqrt{|\det g_{\text{ST}}|}} \partial_\mu \left(\sqrt{|\det g_{\text{ST}}|} F^{\mu\nu} \right)$$

Electrostatics in ST Geometry

Task

Solve Maxwell's equations in ST spacetime $(\mathcal{M}^{n+1}, g_{\text{ST}})$,

$$j^\nu = \frac{1}{\sqrt{|\det g_{\text{ST}}|}} \partial_\mu \left(\sqrt{|\det g_{\text{ST}}|} F^{\mu\nu} \right)$$

Approach

Assuming electrostatic conditions with charge q at position $\mathbf{r}_q$,

$$\Delta_{\mathcal{M}^{n+1}} \Phi(\mathbf{r}, \mathbf{r}_q) = qg_{00}\delta(\mathbf{r} - \mathbf{r}_q) + \frac{r_s^{n-2}(n-2)}{r^{n-1}} \partial_r \Phi(\mathbf{r}, \mathbf{r}_q)$$

Poisson's equation!

Electrostatics in ST Geometry

Task

Solve Maxwell's equations in ST spacetime $(\mathcal{M}^{n+1}, g_{\text{ST}})$,

$$j^\nu = \frac{1}{\sqrt{|\det g_{\text{ST}}|}} \partial_\mu \left(\sqrt{|\det g_{\text{ST}}|} F^{\mu\nu} \right)$$

Approach

Assuming electrostatic conditions with charge q at position $\mathbf{r}_q$,

$$\Delta_{\mathcal{M}^{n+1}} \Phi(\mathbf{r}, \mathbf{r}_q) = q g_{00} \delta(\mathbf{r} - \mathbf{r}_q) + \frac{r_s^{n-2}(n-2)}{r^{n-1}} \partial_r \Phi(\mathbf{r}, \mathbf{r}_q)$$

Poisson's equation! Use multipole ansatz

$$\Phi(\mathbf{r}, \mathbf{r}_q) = \sum_{k \geq 0} R_k(r, r_q) Y_k(\varphi)$$

Radial Equation

Upshot,

$$\lim_{r_q \rightarrow r_s^+} R_k(r, r_q) \propto \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \in \mathbf{Z}^+ \text{ and } (n-2) \mid k \\ \frac{\Gamma(-\frac{2k}{n-2})}{\Gamma(-\frac{k}{n-2})\Gamma(1-\frac{k}{n-2})} & \text{otherwise} \end{cases}$$

Radial Equation

Upshot,

$$\lim_{r_q \rightarrow r_s^+} R_k(r, r_q) \propto \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \in \mathbf{Z}^+ \text{ and } (n-2) \mid k \\ \frac{\Gamma(-\frac{2k}{n-2})}{\Gamma(-\frac{k}{n-2})\Gamma(1-\frac{k}{n-2})} & \text{otherwise} \end{cases}$$

There exist multipole moments (“multipole hair”) when $n > 3$

Radial Equation

Upshot,

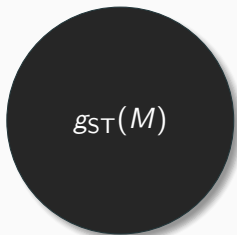
$$\lim_{r_q \rightarrow r_s^+} R_k(r, r_q) \propto \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \in \mathbf{Z}^+ \text{ and } (n-2) \mid k \\ \frac{\Gamma(-\frac{2k}{n-2})}{\Gamma(-\frac{k}{n-2})\Gamma(1-\frac{k}{n-2})} & \text{otherwise} \end{cases}$$

There exist multipole moments (“multipole hair”) when $n > 3$

Hence, final state is not spherically symmetric!

Multipole Hair of ST Black Holes

Schwarzschild-Tangherlini



Something Else
~~Reissner-Nordström-Tangherlini~~



$$(Q \propto q)$$

Final State Properties

ST + Charge Black Hole



Characteristic Properties

- electrically charged

Final State Properties

ST + Charge Black Hole



Characteristic Properties

- electrically charged
- static

Final State Properties

ST + Charge Black Hole



Characteristic Properties

- electrically charged
- static
- asymptotically flat

Final State Properties

ST + Charge Black Hole



Characteristic Properties

- electrically charged
- static
- asymptotically flat
- non-degenerate horizon?

Final State Properties

ST + Charge Black Hole



Characteristic Properties

- electrically charged
- static
- asymptotically flat
- non-degenerate horizon?
- $\mathcal{B} \cong \mathbf{S}^{n-1}$?

Final State Properties

ST + Charge Black Hole



Characteristic Properties

- electrically charged
- static
- asymptotically flat
- non-degenerate horizon?
- $\mathcal{B} \cong \mathbf{S}^{n-1}$?

Final state **not** topologically spherical or possesses degenerate horizon

Acknowledgements

- Jason Gallicchio (HMC)
- Brian Shuve (HMC)
- Brennen Quigley (HMC)
- Thomas Moore (Pomona College)

Radial Equation

$$R_k'' + \frac{n-1}{r} R_k' - \frac{k(k+n-2)r^{n-4}}{r^{n-2} - r_s^{n-2}} R_k = 0$$

Define $\rho = \left(\frac{r_s}{r}\right)^{n-2}$,

$$\rho^2(\rho-1)\ddot{R}_k + \frac{k(k+n-2)}{(n-2)^2} R_k = 0$$

Hypergeometric function! Solve around $\rho = 0$ so $r \in (r_s, +\infty)$:

$$R_k^{(\alpha)}(r, r_s) = \sum_{m \geq 0} \alpha_{k,m} \rho^{m + \chi_k^+}$$

$$R_k^{(\bar{\alpha})}(r, r_s) = \sum_{m \geq 0} \bar{\alpha}_{k,m} \rho^{m - \chi_k^-}$$

Radial Equation

$$R_k^{(\alpha)}(r, r_s) = \sum_{m \geq 0} \alpha_{k,m} \rho^{m + \chi_k^+}$$

$$R_k^{(\bar{\alpha})}(r, r_s) = \sum_{m \geq 0} \bar{\alpha}_{k,m} \rho^{m - \chi_k^-}$$

Here, χ_k^+ and χ_k^- are roots of indicial equation for radial equation,

$$\chi_k^{\pm} = \frac{1 \pm 1}{2} + \frac{k}{n-2}$$

Then,

$$\tilde{\alpha}_{k,m} \equiv \frac{\alpha_{k,m}}{\alpha_{k,0}} = \frac{(\chi_k^+)_m (\chi_k^-)_m}{m! (2\chi_k^+)_m}$$

and

$$\tilde{\bar{\alpha}}_{k,m} \equiv \frac{\bar{\alpha}_{k,m}}{\bar{\alpha}_{k,0}} = \frac{(-\chi_k^-)_m (-\chi_k^+)_m}{m! (-2\chi_k^-)_m}$$

Radial Equation

Matching asymptotic part with electrostatics in $n + 1$ Minkowski,

$$R_k^{(\alpha)}(r, r_s) = r_s^{-(k+n-2)} \sum_{m \geq 0} \tilde{\alpha}_{k,m} \left(\frac{r_s}{r} \right)^{k+(m+1)(n-2)}$$

$$R_k^{(\bar{\alpha})}(r, r_s) = r_s^k \sum_{m \geq 0} \tilde{\tilde{\alpha}}_{k,m} \left(\frac{r}{r_s} \right)^{k-m(n-2)}$$

Green's Function

$$\Phi(\mathbf{r}, \mathbf{r}_q) \propto q \sum_{k \in \Lambda} \sum_{l=1}^{\Gamma_k} \sum_{m \geq 0} \frac{\tilde{\alpha}_{k,m} R_k^{(\bar{\alpha})} Y_k^{*l}(\varphi_q) Y_k^l(\varphi)}{r_s^{k+n-2} (2k+n-2)} \left(\frac{r_s}{r} \right)^{k+(m+1)(n-2)}$$

Here:

$$\Lambda = \{k \in \mathbf{Z}^+ : (n-2) \nmid k\} \cup \{0\}$$

$$\Gamma_k = \frac{(2k+n-2)(n+k-3)!}{k!(n-2)!}$$

There exists a μ_k -pole moment for each $k \in \Lambda$ provided

$$\mu_k \equiv k \pmod{n-2}$$