

Continuous Extensions of Ramanujan-Expandable Arithmetic Functions

arXiv:2010.10689 (*Ramanujan J.*)

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Perfect Numbers: Our Motivation

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- Infinitude? (Euclid-Euler Theorem)
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- (Almost-perfect numbers, quasi-perfect numbers, etc.)

Generalization of perfect numbers

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“What do you mean by the divisors of 3.5, or worse π ?”

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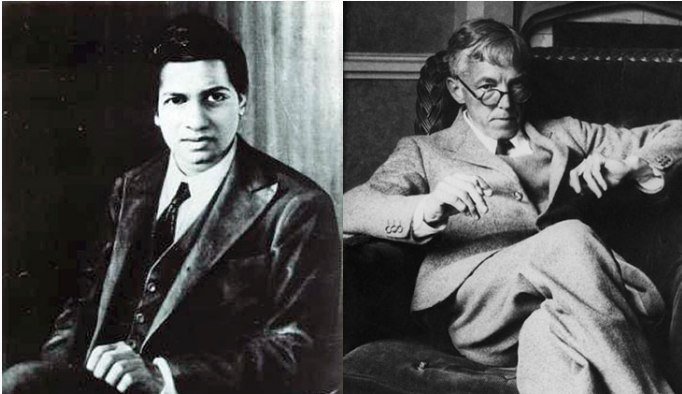
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- Ramanujan's tau function

Circa 1918



Srinivasa Ramanujan and G.H. Hardy were studying the representation of an integer n as the sum of k squares.

G. H. Hardy, *Ramanujan: Twelve Lectures on Subjects Suggested by his Life and Work*. Cambridge, 1940.

Ramanujan Sums and Their Properties

Together, studied properties of one particularly remarkable arithmetic function, now known as the *Ramanujan sum*:

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Orthogonality:

$$\frac{1}{\text{lcm}(q, r)} \sum_{k=1}^{\text{lcm}(q, r)} c_q(k)c_r(k) = \begin{cases} \varphi(q) & q = r \\ 0 & q \neq r. \end{cases}$$

Ramanujan Expansions

Ramanujan proved

$$\sigma_k(n) = n^k \zeta(k+1) \sum_{q \geq 1} \frac{c_q(n)}{q^{k+1}}$$

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Def. An arithmetic function $f : \mathbb{N} \rightarrow \mathbb{C}$ admits a *Ramanujan expansion* iff there exists a sequence $\{\hat{f}(q)\}$ such that $\forall n \in \mathbb{N}$

$$f(n) = \sum_{q \geq 1} \hat{f}(q) c_q(n).$$

Ramanujan Expansions: Famous Examples

Divisor function, $\sigma_k(n)$ where $k > 0$

S. Ramanujan, *Transactions of the Cambridge Phil. Society* **22** (1918), 179–19; G.H. Hardy, *Ramanujan: Twelve Lectures on Subjects Suggested by his Life and Work*. Cambridge, 1940; M. R. Murty, *Hardy-Ramanujan J.* **36** (2013), 21–33

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$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, k \geq 1 \\ 0 & \text{otherwise} \end{cases} = \frac{n}{\varphi(n)} \sum_{q \geq 1} \frac{\mu(q)}{\varphi(q)} c_q(n).$$

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Ramanujan's and Hardy's Zero Functions

$$0_{\text{Ram}}(n) = \sum_{q \geq 1} \frac{c_q(n)}{q} = 0 \quad \text{and} \quad 0_{\text{Har}}(n) = \sum_{q \geq 1} \frac{c_q(n)}{\varphi(q)} = 0$$

Continuous Extensions and Murty's Conjecture

Ramanujan Sums of a Real Variable

Canonically,

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But suppose we modify c_q to some function $\tilde{c}_q$ so that

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Then, get continuous interpolations of all Ramanujan expansions:

$$f(n) = \sum_{q \geq 1} \hat{f}(q) c_q(n) \quad \longrightarrow \quad \tilde{f}(x) := \sum_{q \geq 1} \hat{f}(q) \tilde{c}_q(x),$$

i.e., $\forall n \in \mathbb{N} : \tilde{f}(n) = f(n)$ continuously.

Ramanujan Sums of a Real Variable: Divisor Function

In particular, the divisor function

$$\sigma_k(n) = n^k \zeta(k+1) \sum_{q \geq 1} \frac{c_q(n)}{q^{k+1}} \longrightarrow \tilde{\sigma}_k(x) := x^k \zeta(k+1) \sum_{q \geq 1} \frac{\tilde{c}_q(x)}{q^{k+1}}$$

such that $\forall n \in \mathbb{N} : \tilde{\sigma}_k(n) = \sigma_k(n)$ continuously.

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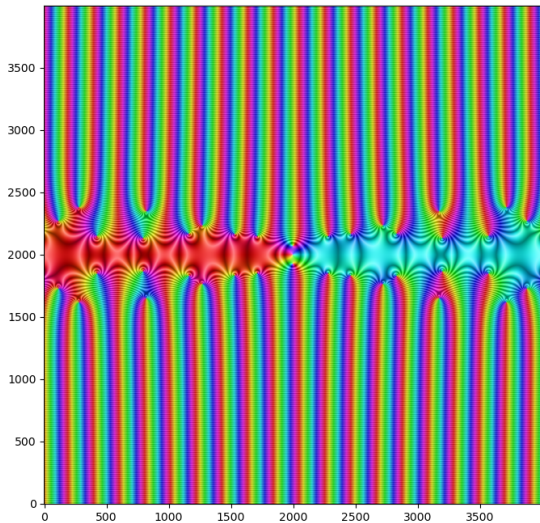
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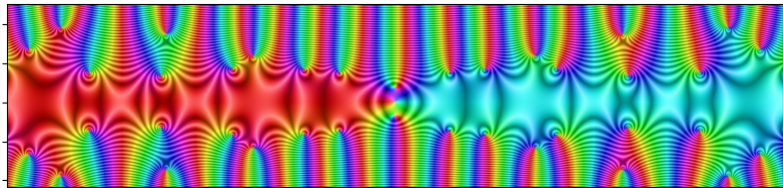
such that $\forall n \in \mathbb{N} : \tilde{\sigma}_k(n) = \sigma_k(n)$ continuously. Then questions about perfect numbers turn into questions about the roots of the function

$$P(x) := \tilde{\sigma}_1(x) - 2x$$

$$\tilde{\sigma}_2(z) - 2z$$



$$\tilde{\sigma}_2(z) - 2z$$



Ramanujan Sums of a Real Variable

Obviously, the Ramanujan sum

$$c_q(n) := \sum_{d|(q,n)} \mu\left(\frac{q}{d}\right) d$$

is not readily amenable to a real variable generalization.

Ramanujan Sums of a Real Variable

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is not readily amenable to a real variable generalization.

However, thanks to a result by Kluuyver,

$$c_q(n) = \sum_{k \in (\mathbb{Z}/q\mathbb{Z})^\times} e^{2\pi i kn/q},$$

which *is* amenable to a real variable generalization.

Ramanujan Sums of a Real Variable

Obvious choice of real variable generalization, $n \rightarrow x$:

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Clearly,

$$\forall n \in \mathbb{N} : \tilde{c}_q(n) = c_q(n)$$

so that, at least naïvely,

$$f(n) = \sum_{q \geq 1} \hat{f}(q) c_q(n) \longrightarrow \tilde{f}(x) := \sum_{q \geq 1} \hat{f}(q) \tilde{c}_q(x)$$

and

$$\forall n \in \mathbb{N} : \tilde{f}(n) = f(n).$$

Ramanujan Sums of a Real Variable: Murty

This idea was proposed by Murty in 2013, but left unexplored.

$$f(n) = \sum_{q \geq 1} \hat{f}(q) c_q(n) \quad \longrightarrow \quad \tilde{f}(x) := \sum_{q \geq 1} \hat{f}(q) \tilde{c}_q(x)$$

Murty: “The interesting thing... is that if the series converges absolutely and we replace n by a real number x , we obtain a continuous function which interpolates the given arithmetical function. In this way, we can view the Ramanujan expansion as a continuous analogue of the discretely defined arithmetical function.”

Ramanujan Sums of a Real Variable: Heuristics

For this choice of $\tilde{c}_q : \mathbb{R} \rightarrow \mathbb{C}$,

$$\begin{aligned}\tilde{c}_q(x) &= \sum_{k \in (\mathbb{Z}/q\mathbb{Z})^\times} e^{2\pi i k x / q} \\ &= \sum_{k \in (\mathbb{Z}/q\mathbb{Z})^\times} \operatorname{Re}(e^{2\pi i k x / q}) + i \sum_{k \in (\mathbb{Z}/q\mathbb{Z})^\times} \operatorname{Im}(e^{2\pi i k x / q}).\end{aligned}$$

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For “arbitrary” q and each k ,

$$\begin{aligned}R_q(k) &:= \operatorname{Re}(e^{2\pi i k x / q}) \sim \operatorname{Uniform}(-1, 1) \\ I_q(k) &:= \operatorname{Im}(e^{2\pi i k x / q}) \sim \operatorname{Uniform}(-1, 1).\end{aligned}$$

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Then, roughly speaking,

$$\tilde{c}_q(x) \approx \mathbb{E} \left[\sum_{k \in (\mathbb{Z}/q\mathbb{Z})^\times} (R_q(k) + iI_q(k)) \right] = \sum_{k \in (\mathbb{Z}/q\mathbb{Z})^\times} (0) = 0.$$

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it is plain that

$$\begin{aligned} |c_q(n)| &\leq \sum_{d|(q,n)} \left| \mu\left(\frac{q}{d}\right) \cdot d \right| \\ &\leq \sum_{d|n} \left| \mu\left(\frac{q}{d}\right) \cdot d \right| \\ &\leq \sum_{d|n} |d| \\ &= \sigma_1(n) \in O_n(1). \end{aligned}$$

Ramanujan Sums of a Real Variable: Conjectures

Let

$$f(n) = \sum_{q \geq 1} \widehat{f}(q) c_q(n)$$

and suppose we want f to absolutely converge,

$$\sum_{q \geq 1} \left| \widehat{f}(q) c_q(n) \right| < \infty.$$

Since $|c_q(n)| = O_n(1)$, asymptotically we must have

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Lemma: Let $f : \mathbb{N} \rightarrow \mathbb{N}$ admit a Ramanujan expansion with Ramanujan coefficients $\{\widehat{f}(q) : q \geq 1\}$. If $\widehat{f}(q) = o(q^{-1})$, then the Ramanujan expansion of f absolutely converges.

Ramanujan Sums of a Real Variable: Conjectures

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Murty's Conjecture (MC): If $f : \mathbb{N} \rightarrow \mathbb{N}$ admits an absolutely convergent Ramanujan expansion, then its real variable generalization, $\tilde{f} : \mathbb{R} \rightarrow \mathbb{C}$, is continuous (and hence converges).

That is,

$$\sum_{q \geq 1} \left| \hat{f}(q) c_q(n) \right| < \infty \implies \left| \sum_{q \geq 1} \hat{f}(q) \tilde{c}_q(x) \right| < \infty$$

Asymptotics of $\tilde{c}_q(x)$ and $\tilde{\sigma}_k(x)$

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Summary of logic:

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$$\widehat{f}(q) = o(q^{-1}) \xRightarrow{\text{Lemma}} \sum_{q \geq 1} \left| \widehat{f}(q) c_q(n) \right| < \infty \xRightarrow{\text{MC}} \left| \sum_{q \geq 1} \widehat{f}(q) \widetilde{c}_q(x) \right| < \infty.$$

Ramanujan Sums of a Real Variable: Conjectures

Summary of logic:

$$\widehat{f}(q) = o(q^{-1}) \xRightarrow{\text{Lemma}} \sum_{q \geq 1} \left| \widehat{f}(q) c_q(n) \right| < \infty \xRightarrow{\text{MC}} \left| \sum_{q \geq 1} \widehat{f}(q) \widetilde{c}_q(x) \right| < \infty.$$

Thus, if there exists some $\widehat{f}(q)$ such that

$$\widehat{f}(q) = o(q^{-1}) \implies \left| \sum_{q \geq 1} \widehat{f}(q) \widetilde{c}_q(x) \right| = \infty,$$

then **MC** is false.

Ramanujan Sums of a Real Variable: Conjectures

A hint that **MC** is false:

$$\tilde{\sigma}_1(x) = x\zeta(2) \sum_{q \geq 1} \frac{\tilde{c}_q(x)}{q^2}.$$

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Obviously $q^{-2} = o(q^{-1})$, but

$$\sum_{q \geq 1} \frac{\tilde{c}_q(0)}{q^2} = \sum_{q \geq 1} \frac{\varphi(q)}{q^2} = \lim_{k \rightarrow 1^+} \frac{\zeta(k)}{\zeta(k+1)} = \infty.$$

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Though, $x = 0$ is pathological *in general* ($\tilde{c}_q(0) = c_q(0) = \varphi(q)$).
Any novel pathologies ($x \in \mathbb{R} \setminus \mathbb{Q}$)?

Ramanujan Sums of a Real Variable: Results

Theorem: For fixed $x \in \mathbb{R} \setminus \mathbb{Q}$ and all $q > 2\pi|x|$,

$$\tilde{c}_q(x) = \frac{e^{2\pi ix} - 1}{2\pi ix} \varphi(q) + O_x(\sigma_0(q)).$$

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Corollary 1: If $f : \mathbb{N} \rightarrow \mathbb{N}$ admits a Ramanujan expansion with positive Ramanujan coefficients $\hat{f} = \Omega(q^{-2})$, then for all $x \in \mathbb{R} \setminus \mathbb{Q}$, its continuous generalization $\tilde{f}$ diverges.

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Corollary 2: Murty's Conjecture (MC) is false.

Corollary 3: The divisor function

$$\sigma_k(n) := \sum_{d|n} d^k = n^k \zeta(k+1) \sum_{q \geq 1} \frac{c_q(n)}{q^{k+1}}$$

does not continuously extend to $\mathbb{R}$ if $0 \leq k \leq 1$ (but does for all $k > 1$).

Ramanujan Sums of a Real Variable: $\tilde{c}_q(x) = O_x(\varphi(q))$

Proof of Theorem (sketch):

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \implies \tilde{c}_q(x) = \sum_{d|q} \mu(d) \sum_{l=1}^{q/d} e^{2\pi i l d x / q}.$$

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$x \in \mathbb{R} \setminus \mathbb{Q}$ allows us to use geometric sum (and not worry about $q \rightarrow \infty$):

$$\tilde{c}_q(x) = (1 - e^{2\pi i x}) \sum_{d|q} \frac{\mu(q/d)}{e^{-2\pi i x/d} - 1}.$$

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Ramanujan Sums of a Real Variable: $\tilde{c}_q(x) = O_x(\varphi(q))$

Proof of Theorem (sketch):

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \implies \tilde{c}_q(x) = \sum_{d|q} \mu(d) \sum_{l=1}^{q/d} e^{2\pi i l d x / q}.$$

$x \in \mathbb{R} \setminus \mathbb{Q}$ allows us to use geometric sum (and not worry about $q \rightarrow \infty$):

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Möbius inversion, $q = \varphi * 1 \implies \varphi = q * \mu$, implies

$$\sum_{d|q} \mu\left(\frac{q}{d}\right) d = \varphi(q) \implies \sum_{\substack{d|q \\ d > 2\pi|x|}} \mu\left(\frac{q}{d}\right) d = \varphi(q) + O_x(1).$$

Thus,

$$\sum_{\substack{d|q \\ d > 2\pi|x|}} \mu\left(\frac{q}{d}\right) \left(-\frac{d}{2\pi i x} + O_x(1)\right) = -\frac{\varphi(q)}{2\pi i x} + O_x(\sigma_0(q))$$

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Altogether,

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Corollary 1: If $f : \mathbb{N} \rightarrow \mathbb{N}$ admits a Ramanujan expansion with positive Ramanujan coefficients $\hat{f} = \Omega(q^{-2})$, then for all $x \in \mathbb{R} \setminus \mathbb{Q}$, its continuous generalization $\tilde{f}$ diverges.

Ramanujan Sums of a Real Variable: Ramanujan Expansions

Proof of Corollary 1:

$$\tilde{f}(x) = \sum_{q \geq 1} \hat{f}(q) \tilde{c}_q(x)$$

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Since $x \in \mathbb{R} \setminus \mathbb{Q}$ and $q > 2\pi|x|$, by the **Theorem**

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By $\hat{f}(q) = \Omega(q^{-2})$ and positivity, $\hat{f}(q) \geq Cq^{-2}$, so

$$\sum_{q > 2\pi|x|} \hat{f}(q) \varphi(q) \geq C \sum_{q \geq 1} \frac{\varphi(q)}{q^2} + O_x(1) = \infty.$$

□

Ramanujan Sums of a Real Variable: Ramanujan Expansions

Murty's Conjecture (MC): If $f : \mathbb{N} \rightarrow \mathbb{N}$ admits an absolutely convergent Ramanujan expansion, then its real variable generalization, $\tilde{f} : \mathbb{R} \rightarrow \mathbb{C}$, is continuous (and hence converges).

Theorem: For fixed $x \in \mathbb{R} \setminus \mathbb{Q}$ and all $q > 2\pi|x|$,

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Corollary 2: MC is false.

Proof: $q^{-2} \in \Omega(q^{-2}) \cap o(q^{-1})$, so **Corollary 1** implies MC is false.

Ramanujan Sums of a Real Variable: $\tilde{\sigma}_k(x)$ and $\sigma_1(n)$

Corollary 3: The generalized divisor function

$$\tilde{\sigma}_k(x) := x^k \zeta(k+1) \sum_{q \geq 1} \frac{\tilde{c}_q(x)}{q^{k+1}}$$

- converges absolutely for all $x \in \mathbb{Z}$ if $k > 0$,
- converges absolutely for all $x \in \mathbb{R}$ if $k > 1$,
- *diverges* for all $x \in \mathbb{R} \setminus \mathbb{Q}$ if $0 \leq k \leq 1$.

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In particular,

$$\sigma_1(n) = \sum_{d|n} d = n \zeta(2) \sum_{q \geq 1} \frac{c_q(n)}{q^2}$$

does not continuously extend to $\mathbb{R}$.

Ramanujan Sums of a Real Variable: Divergent Expansions

Ramanujan's and Hardy's zero functions

$$\tilde{0}_{\text{Ram}}(x) := \sum_{q \geq 1} \frac{\tilde{c}_q(x)}{q} \quad \text{and} \quad \tilde{0}_{\text{Har}}(x) := \sum_{q \geq 1} \frac{\tilde{c}_q(x)}{\varphi(q)}.$$

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Number-of-divisors function

$$\tilde{\sigma}_0(x) := - \sum_{q \geq 1} \frac{\log q}{q} \tilde{c}_q(x).$$

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Most interesting (at least to us) is failure of extension for $\sigma_1(n)$,

$$\tilde{\sigma}_1(x) = x\zeta(2) \sum_{q \geq 1} \frac{\tilde{c}_q(x)}{q^2}.$$

Question: Does there exist a continuous extension of σ_1 ?

A (Not-So-Exciting) Continuation of $\sigma_1(n)$

Idea: $\tilde{\sigma}_k(x)$ converges for all $x \in \mathbb{R}$ if $k > 1$. Find M, A_k such that

$$\tilde{\sigma}_1(x) := \sum_{k=2}^M A_k \tilde{\sigma}_k(x).$$

(An “impure” Ramanujan expansion.)

A (Not-So-Exciting) Continuation of $\sigma_1(n)$

For $\alpha \in \mathbb{Z} \setminus \{0\}$ with divisors $d_1, \dots, d_N$, where $N = \sigma_0(\alpha)$, define

$$P_\alpha(x) := \prod_{i=1}^N (x - d_i) = \sum_{k=0}^N a_n x^n.$$

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Newton's sums approach:

$$\begin{cases} P_\alpha(d_1) = 0 \\ P_\alpha(d_2) = 0 \\ \vdots \\ P_\alpha(d_N) = 0 \end{cases} \implies \begin{cases} a_N d_1^{k+N} + \dots + a_1 d_1^{k+1} + a_0 d_1^k = 0 \\ a_N d_2^{k+N} + \dots + a_1 d_2^{k+1} + a_0 d_2^k = 0 \\ \vdots \\ a_N d_N^{k+N} + \dots + a_1 d_N^{k+1} + a_0 d_N^k = 0 \end{cases}$$

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Adding terms,

$$a_N \underbrace{(d_1^{k+N} + \dots + d_N^{k+N})}_{\sigma_{k+N}(\alpha)} + \dots + a_0 \underbrace{(d_1^k + \dots + d_N^k)}_{\sigma_k(\alpha)} = 0$$

A (Not-So-Exciting) Continuation of $\sigma_1(n)$

Thus,

$$\sigma_k(\alpha) = -\frac{1}{a_0} (a_N \sigma_{k+N}(\alpha) + \cdots + a_1 \sigma_{k+1}(\alpha)).$$

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Ultimately, this defines an impure Ramanujan expansion for $\sigma_1(n)$:

$$\sigma_1(\alpha) = \sum_{q \geq 1} \sum_{j=0}^{N-1} \left[\frac{(-1)^{j+1} e_j(d_1, \dots, d_N)}{d_1 \cdots d_N} \cdot \frac{\zeta(N+2-j) \alpha^{N+1-j}}{q^{N+2-j}} \right] c_q(\alpha).$$

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Get continuous impure Ramanujan expansion by taking $c_q \rightarrow \tilde{c}_q$:

$$\tilde{\sigma}_1(x, \alpha) := \sum_{q \geq 1} \sum_{j=0}^{N-1} \left[\frac{(-1)^{j+1} e_j(d_1, \dots, d_N)}{d_1 \cdots d_N} \cdot \frac{\zeta(N+2-j) x^{N+1-j}}{q^{N+2-j}} \right] \tilde{c}_q(x).$$

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Obviously fails, for only guaranteed to interpolate on $(\alpha - 1, \alpha + 1)$.

Where to next?

Other Ideas: What causes $\tilde{\sigma}_1$ to diverge?

Theorem: For fixed $x \in \mathbb{R} \setminus \mathbb{Q}$ and all $q > 2\pi|x|$,

$$\tilde{c}_q(x) = \frac{e^{2\pi i x} - 1}{2\pi i x} \varphi(q) + O_x(\sigma_0(q)).$$

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$$\begin{aligned} \tilde{c}_q(x) = & \frac{e^{2\pi i x} - 1}{2\pi i x} \varphi(q) + \frac{e^{2\pi i x} - 1}{2} \left[\frac{1}{q} \right] \\ & + \sum_{t \geq 2} \frac{(2\pi i x)^t}{(t+1)!} \sum_{k=2}^t \binom{t+1}{k} B_k \sum_{d|q} \mu(q/d) d^{1-k} \end{aligned}$$

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If $s > 1$, then

$$\sigma_s(x) = x^s \sum_{t \geq 2} \frac{(2\pi i x)^t}{(t+1)!} \sum_{k=1}^{t-1} \binom{t+1}{k+1} B_{k+1} \zeta(k+s+1)$$

Other Ideas: Ramanujan Tau Function

The Ramanujan Tau function, $\tau : \mathbb{N} \rightarrow \mathbb{Z}$, is defined as

$$\Delta(z) = \sum_{n \geq 1} \tau(n) e^{2\pi i z}$$

where $\Delta(z)$ is the modular discriminant.

Other Ideas: Ramanujan Tau Function

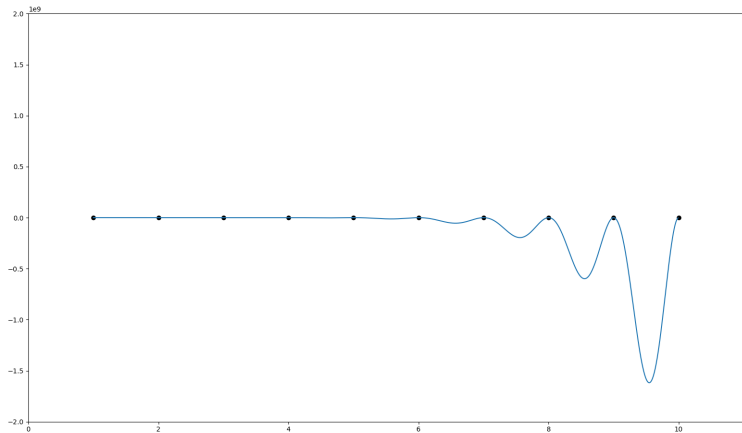
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where $\Delta(z)$ is the modular discriminant.

- $\tau(1) = 1$
- $\tau(10) = -115920$
- $\tau(15) = 1217160$

Interpolation of $\tau(n)$ (real part)



Thank you!

Proof of Theorem

Ramanujan Sums of a Real Variable: $\tilde{c}_q(x) = O_x(\varphi(q))$

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$$\tilde{c}_q(x) = \frac{e^{2\pi i x} - 1}{2\pi i x} \varphi(q) + O_x(\sigma_0(q)).$$

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$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \implies \tilde{c}_q(x) = \sum_{k=1}^q e^{2\pi i kx/q} \sum_{d|(k,q)} \mu(d).$$

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Swapping order of summation,

$$\tilde{c}_q(x) = \sum_{d|q} \mu(d) \sum_{l=1}^{q/d} e^{2\pi i ldx/q}.$$

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Swapping order of summation,

$$\tilde{c}_q(x) = \sum_{d|q} \mu(d) \sum_{l=1}^{q/d} e^{2\pi i l d x / q}.$$

But for all $x \in \mathbb{R} \setminus \mathbb{Q}$, $|e^{2\pi i l d x / q}| < 1$, so

$$\tilde{c}_q(x) = (1 - e^{2\pi i x}) \sum_{d|q} \frac{\mu(q/d)}{e^{-2\pi i x / d} - 1}.$$

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Since $q > 2\pi|x|$,

$$\begin{aligned}\frac{\tilde{c}_q(x)}{(1 - e^{2\pi ix})} &= \sum_{d|q} \frac{\mu(q/d)}{e^{-2\pi ix/d} - 1} \\ &= \sum_{\substack{d|q \\ d < 2\pi|x|}} \frac{\mu(q/d)}{e^{-2\pi ix/d} - 1} + \sum_{\substack{d|q \\ d > 2\pi|x|}} \frac{\mu(q/d)}{e^{-2\pi ix/d} - 1}\end{aligned}$$

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Notice, for the first sum

$$\left| \sum_{\substack{d|q \\ d < 2\pi|x|}} \frac{\mu(q/d)}{e^{-2\pi ix/d} - 1} \right| \leq \sum_{j=1}^{\lfloor 2\pi|x| \rfloor} \left| \frac{1}{e^{-2\pi ix/j} - 1} \right| = O_x(1).$$

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Thus,

$$\frac{\tilde{c}_q(x)}{(1 - e^{2\pi ix})} = O_x(1) + \sum_{\substack{d|q \\ d > 2\pi|x|}} \frac{\mu(q/d)}{e^{-2\pi ix/d} - 1}.$$

Ramanujan Sums of a Real Variable: $\tilde{c}_q(x) = O_x(\varphi(q))$

For the second sum, by Taylor expansion of the exponential and $d > 2\pi|x|$,

$$\sum_{\substack{d|q \\ d > 2\pi|x|}} \frac{\mu(q/d)}{e^{-2\pi ix/d} - 1} = -\frac{1}{2\pi ix} \sum_{\substack{d|q \\ d > 2\pi|x|}} \mu\left(\frac{q}{d}\right) d + \sum_{\substack{d|q \\ d > 2\pi|x|}} \mu\left(\frac{q}{d}\right) O_x(1).$$

Ramanujan Sums of a Real Variable: $\tilde{c}_q(x) = O_x(\varphi(q))$

For the second sum, by Taylor expansion of the exponential and $d > 2\pi|x|$,

$$\sum_{\substack{d|q \\ d > 2\pi|x|}} \frac{\mu(q/d)}{e^{-2\pi ix/d} - 1} = -\frac{1}{2\pi ix} \sum_{\substack{d|q \\ d > 2\pi|x|}} \mu\left(\frac{q}{d}\right) d + \sum_{\substack{d|q \\ d > 2\pi|x|}} \mu\left(\frac{q}{d}\right) O_x(1).$$

But Möbius inversion of Gauss' sum $\sum_{d|q} \varphi(d) = q$ gives $\varphi(q) = \sum_{d|q} \mu(q/d)d$. So,

$$\sum_{\substack{d|q \\ d > 2\pi|x|}} \mu\left(\frac{q}{d}\right) d = \varphi(q) - \sum_{\substack{d|q \\ d < 2\pi|x|}} \mu\left(\frac{q}{d}\right) d = \varphi(q) + O_x(1).$$

Ramanujan Sums of a Real Variable: $\tilde{c}_q(x) = O_x(\varphi(q))$

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Similarly,

$$\sum_{\substack{d|q \\ d > 2\pi|x|}} \mu\left(\frac{q}{d}\right) O_x(1) = O_x(\sigma_0(q)).$$

Ramanujan Sums of a Real Variable: $\tilde{c}_q(x) = O_x(\varphi(q))$

Altogether,

$$\frac{\tilde{c}_q(x)}{(1 - e^{2\pi ix})} = \underbrace{\sum_{\substack{d|q \\ d < 2\pi|x|}} \frac{\mu(q/d)}{e^{-2\pi ix/d} - 1}}_{O_x(1)} + \underbrace{\sum_{\substack{d|q \\ d > 2\pi|x|}} \frac{\mu(q/d)}{e^{-2\pi ix/d} - 1}}_{-\frac{1}{2\pi ix} \varphi(q) + O_x(\sigma_0(q))}.$$

Hence, for fixed $x \in \mathbb{R} \setminus \mathbb{Q}$ and $q > 2\pi|x|$,

$$\tilde{c}_q(x) = \frac{e^{2\pi ix} - 1}{2\pi ix} \varphi(q) + O_x(\sigma_0(q)).$$

Ramanujan Sums of a Real Variable: $\tilde{c}_q(x) = O_x(\varphi(q))$

Altogether,

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So, as desired, we have proven

Theorem: For fixed $x \in \mathbb{R} \setminus \mathbb{Q}$ and $q > 2\pi|x|$, then

$$\tilde{c}_q(x) = \frac{e^{2\pi ix} - 1}{2\pi ix} \varphi(q) + O_x(\sigma_0(q)).$$