# Some Things on Bipartite Entanglement

## AGENDA:

- TPS's, Separable States, the QSP
- EPR, Bell, Tsierlson
- LOCC, Nielsen's theorem, MoE

## PRIMARY REFs:

- Nielsen & Chuang
- quant-ph/0702225 (Entanglement review)
- 1702.06142 (TPS)
- 1210.4583 (LOCC)

## THEOREM REFs:

- quant-ph/0702225 (HJW and Carathéodory)
- quant-ph/0303055 (Gurvitz)
- 1802.10119 (Bell)
- 2001.04383 (Ji et al.)
- quant-ph/9811053 (Nielsen)
- quant-ph/9707038 (Lo & Popescu)
- quant-ph/9807077 (Vidal)

## 0. Notation & Conventions.

For vectors (a.k.a. pure states) $|a_i\rangle \in \mathcal{H}$,

$$|a_1\rangle|a_2\rangle \cdots := |a_1\rangle \otimes |a_2\rangle \otimes \cdots$$

For example, the Bell states are

$$|\Phi^+\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$$
$$|\Phi^-\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle - |1\rangle|1\rangle\right)$$
$$|\psi^+\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + |1\rangle|0\rangle\right)$$
$$|\psi^-\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right)$$

However, for states $\rho_i \in \mathcal{D}(\mathcal{H}_i)$,

$$\rho_i\rho_j \neq \rho_i \otimes \rho_j$$

Recall,

$$\rho \in \mathcal{D}(\mathcal{H}) \iff \begin{array}{ll} \text{(i) } \forall|\psi\rangle \in \mathcal{H}: \langle\psi|\rho|\psi\rangle \geq 0 & \text{(Positive semi-definite)} \\ \text{(ii) } \rho^\dagger = \rho & \text{(Hermitian)} \\ \text{(iii) } \mathrm{Tr}(\rho) = 1 & \text{(unit trace)} \end{array}$$

<u>Lemma</u>: $\mathcal{D}(\mathcal{H})$ is a convex set. That is, $\rho_1, \rho_2 \in \mathcal{D}(\mathcal{H})$ and $\lambda \in [0,1]$ implies

$$\lambda\rho_1 + (1-\lambda)\rho_2 \in \mathcal{D}(\mathcal{H}).$$

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# 1. Tensor Product Structures (TPSs).

... Transportation & parking services

Let $|\psi\rangle = \alpha^i |e_i\rangle$, and $\rho = |\psi\rangle\langle\psi|$.

Q: Is $\rho$ entangled?

- No! This question makes no sense.
- Entanglement is always with respect to something.
- One therefore needs additional structure.

Def. (rough): A TPS on $\mathcal{H}$ is a preferred factorization:

$$\mathcal{H} = \bigotimes_i \mathcal{H}_i .$$

- A TPS is put in by hand.
- $\mathcal{H}$ does not come pre-equipped w/ "best" TPS.
- Thus, every TPS demands justification.
- In quantum computing/information, spatial locality usually determines the TPS.

Def. A TPS is bipartite iff

$$\mathcal{H} = \underset{\text{"Alice"}}{\underset{\uparrow}{\mathcal{H}_A}} \otimes \underset{\text{"Bob"}}{\underset{\uparrow}{\mathcal{H}_B}} =: \mathcal{H}_{AB} .$$

<u>Notation:</u> $D_{AB} := \mathcal{D}(\mathcal{H}_{AB})$, $d_{AB} = \dim(\mathcal{H}_{AB})$.

- Hereafter assume $d_{AB} < \infty$ (except Tsierlson)

Every $\rho \in D_{AB}$ Satisfies:

$$\rho = P_i \rho^i \qquad \text{(spectral decomp.)}$$

$$= P_{imn} \, \sigma_A^{im} \otimes \sigma_B^{in} \qquad \text{(bipart-ness)}$$

Hughston Jozsa Wooters

<u>Thm. (HJW):</u> Not unique. ("Different ways to realize any given ensemble")

- Penington et al. article.

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# 2. Separable and Entangled States

**Def.** A bipartite state $\rho \in \mathcal{D}_{AB}$ is <u>Separable</u> iff for some $K$,

$$\rho = \sum_{i=1}^{K} p_i \, \sigma_A^i \otimes \sigma_B^i .$$

... "Convex sum of Product States"

Conversely, $\rho$ is <u>entangled</u> iff $\rho$ is not separable.

**Remark:** WLOG, $\sigma_A^i$ and $\sigma_B^i$ are pure.

- Does this def. correspond to the familiar one?

**Prop.** $|\psi\rangle \in \mathcal{H}_{AB}$ is separable iff

$$|\psi\rangle = |\psi_A\rangle|\psi_B\rangle .$$

**Q:** What is $K$?

**Thm. (Carathéodory):** $K \leq d_{AB}^2$.

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## 3. Deciding Entanglement.

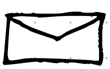
Let

$$S_{AB} := \{\rho \in \mathcal{D}_{AB} \mid \rho \text{ separable}\}.$$

**Def.** The <u>quantum separability problem</u> (QSP) is to compute the indicator function

$$\mathbb{1}_{AB}(\rho) := \begin{cases} 1 & \text{if } \rho \in S_{AB}, \\ 0 & \text{otherwise}. \end{cases}$$

Q: How hard is this?

 :

- For any collection $\{\sigma_A^i \otimes \sigma_B^i\}_{i=1}^{k} \subseteq \mathcal{D}(\mathcal{H}_{AB})$, $\exists\, 2^k$ subsets.
- $k \leq d_{AB}^2$ (Carathéodory) $\Rightarrow$ time $= O(2^{d_{AB}^2})$.
- Need $n = \log_2 d_{AB}$ qubits to build $\mathcal{H}_{AB}$
- So, to decide QSP is like $O(2^{2^{2n}})$ computational steps!
- For $n=4$ qubits, that's

$$2^{2^8} \simeq 10^{77} \text{ steps!}$$

- Can get super-exponential increase for pure states.

<u>Def./Thm.:</u> The <u>Schmidt decomposition</u> of $|\psi\rangle \in \mathcal{H}_{AB}$ is

$$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |i_A\rangle |i_B\rangle,$$

where $n \leq \min\{d_A, d_B\}$ and $\lambda_i$ are the <u>Schmidt coefficients</u>, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i^2 = 1$.

- This does not generally extend to multipartite setting.

<u>Lem:</u> Let $|\psi\rangle \in \mathcal{H}_{AB}$. Then:

(i) $|\psi\rangle$ separable iff $\lambda_i = 1$ and $\lambda_j = 0$ $\forall j \neq i$.

(ii)

spectral decomposition
$$\begin{cases} \rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \sum_{i=1}^{n} \lambda_i^2 |u_i\rangle\langle u_i|, \\ \rho_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|) = \sum_{i=1}^{n} \lambda_i^2 |v_i\rangle\langle v_i|. \end{cases}$$

(iii) $A \in \mathbb{R}^{n \times n}$, E-VALS$(A) \in$ BPP

"Fox entanglement test".

<u>Thm:</u> $QSP^{\text{Pure}} \in BPP$.

However,

<u>Thm. (Gurvitz):</u> In most cases, QSP $\in$ NP-Hard.

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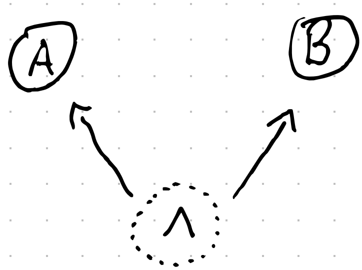
# 4. EPR & Bell.

- Consider Alice and Bob Spacelike Separated.
- What joint Probability distributions Can they generate?

<u>$P(A,B)$:</u>

$S_1$:  $\cong$ $P(A)P(B)$.

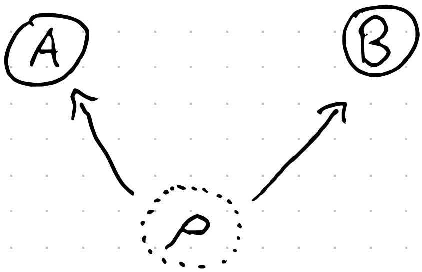
$S_2$:  $\cong$ $\sum_{\lambda \in \Lambda} P(A|\lambda)P(B|\lambda)P(\lambda)$

<u>Notation</u>: $\mathcal{D}(S)$ = "Set of Probability distributions Compatible with Situation $S$".

Q: $\mathcal{D}(S_1) = \mathcal{D}(S_2)$ ?

No! Correlations Can exist in $S_2$.

$S_3$:




$$\cong \mathrm{Tr}(M_A \otimes M_B \, \rho)$$

<u>Q</u>: $\mathcal{D}(S_2) = \mathcal{D}(S_3)$?

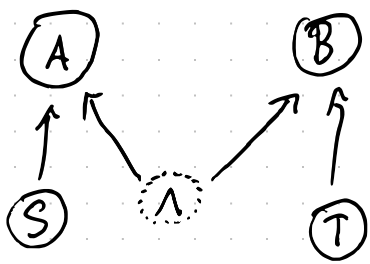
Yes! Provided $\Lambda$ is "big enough" to simulate correlations.

So,

"EPR Paradox" $\begin{cases} \text{Operationally} : & S_2 = S_3 \\ \text{Metaphysically} : & S_2 \neq S_3 \quad \text{(entanglement!)} \end{cases}$

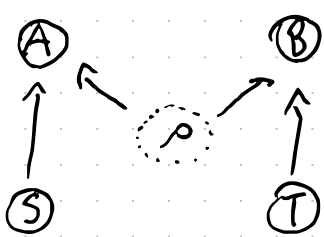
- To really see what's up, consider following scenarios: <u>$P(A,B \mid S,T)$:</u>

$S_4$:




$$\cong \sum_{\lambda \in \Lambda} P(A \mid S,\lambda)\, P(B \mid T,\lambda) \times P(S)\, P(T)\, P(\lambda).$$

$S_5$:




$$\cong \mathrm{Tr}(M_{A|S} \otimes M_{B|T} \, \rho).$$

Q: $\mathcal{D}(S_4) = \mathcal{D}(S_5)$ ?

Thm. (Bell): NO!

Proof: $\mathcal{D}(S_4)$ = Semialgebraic Set

$\Rightarrow$ suffices to find $P \in \mathcal{D}(S_5)$ that violates one of the inequalities.

- So, entanglement allows A and B to generate a larger set of correlations.

- Very obviously important for information processing!

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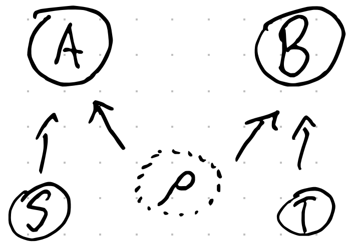
# 5. Tsirelson's Problem.

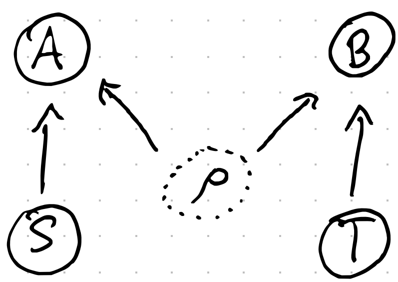
$M_{A|S} \in POVM(\mathcal{H}_A)$
$M_{B|T} \in POVM(\mathcal{H}_B)$

$S_5$:




$$\cong \mathrm{Tr}(M_{A|S} \otimes M_{B|T}\, \rho)$$

$S_5'$:




$$\cong \mathrm{Tr}(M_{A|S} M_{B|T}\, \rho),$$

where

$$M_{A|S}, M_{B|T} \in POVM(\mathcal{H}_{AB}),$$

$$[M_{A|S}, M_{B|T}] = 0 .$$

Tsirelson's Problem: $\mathcal{D}(S_5) = \mathcal{D}(S_5')$ ?

- Evidently, $\mathcal{D}(S_5) \subseteq \mathcal{D}(S_5')$. But what about the other way?

Thm. (Ji et al.): No!

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# 6. Some Quantum Protocols.

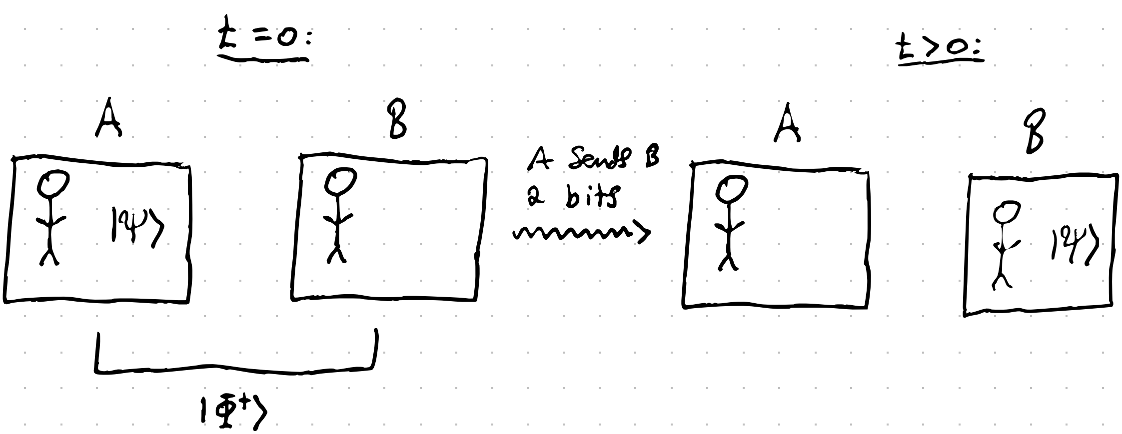
- The last few Sections Prove that entanglement may be useful for information Processing.

**Q:** How many bits does a qubit $|\psi\rangle$ store?

Infinite? Finite?

- If you think more than two then the following Protocol is surprising:

## Quantum Teleportation:



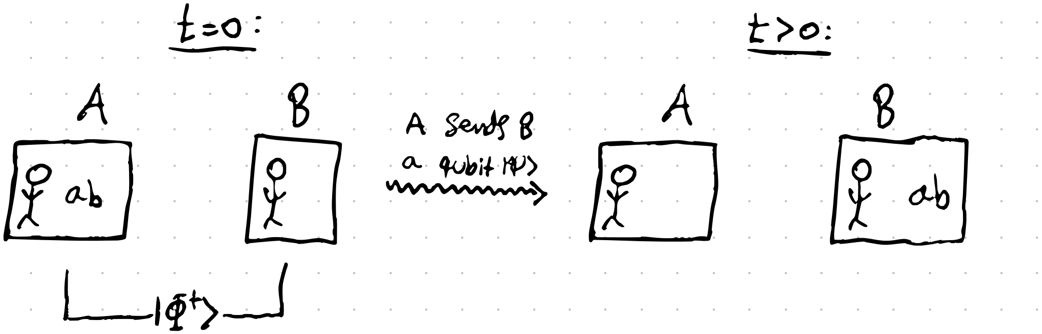

- For brevity, not going to go through Protocol. I want to instead mention two others that are interesting.
- But first:

**Remark:**
- Teleportation $\not\Rightarrow$ quantum advantage (Gottesman-Knill thm.)
- Teleportation is Possible in Classical theories (Spekkens Toy Model)

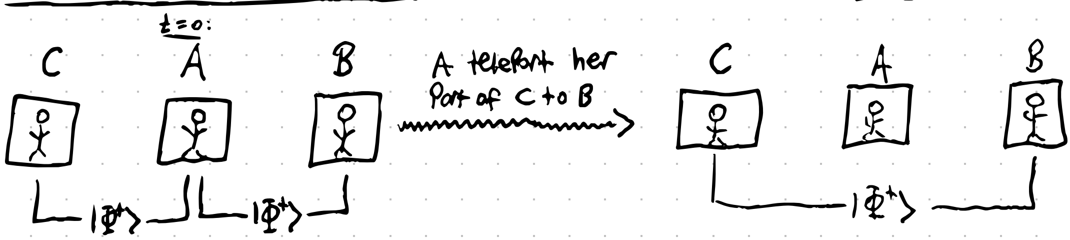
## Superdense Coding:

- Opposite teleportation:

Let $ab \in \{00, 01, 10, 11\}$. Then:




## Entanglement Swapping:




Moral: Entanglement can be created w/o local interaction.

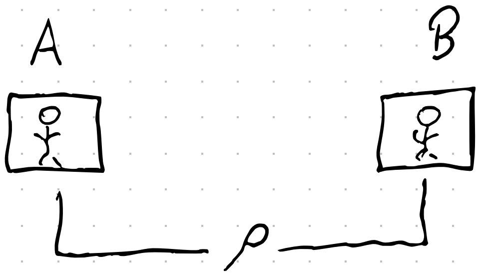
## Quantum Energy Teleportation (QET).

- Borne out of the relatively nascent field RQI.
- Essentially combines Landauer's principle w/ teleportation.

See 1101.3954 (good review by Hotta).

# 7. LOCC.

- Let us treat entanglement like a valuable commodity of which we only have so much.



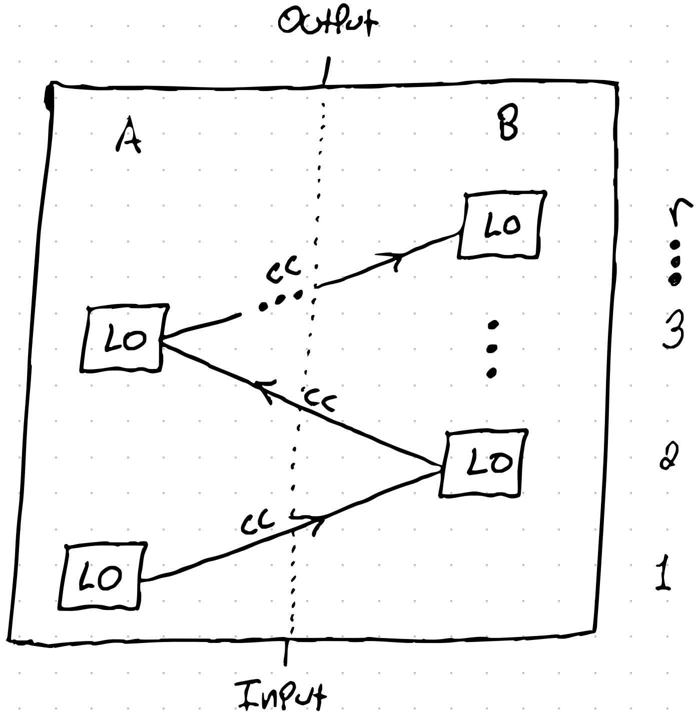

Q: What operations between A and B do not "increase" entanglement?

(i) LO

(ii) CC

- Practical restriction — nature does not care about CC.

**Def.** $\mathcal{E} \in CPTP(\mathcal{H}_{AB})$ is <u>r-round LOCC Protocol</u> iff:

$\mathcal{E} =$ 


**Def.** $(\rho, \sigma) \in LOCC_r$ iff $\exists$ r-round $\mathcal{E}$ s.t.

$\mathcal{E}(\rho) = \sigma$.

**Thm. (Lo & Popescu):**

$$LOCC^{pure, bi} := \bigcup_{r \geq 1} LOCC_r^{pure, bi} = LOCC_2^{pure, bi}.$$

- That is why teleportation is the way it is.
- Does not hold for mixed or multipartite states.
- LOCC article.

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# 8. Nielsen's Theorem.

Let $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Define

$x_i^{\downarrow}$ := the ith largest component of $\vec{x}$.

**Def.** For $\vec{x}, \vec{y} \in \mathbb{R}^n$, we say $\vec{x}$ majorizes $\vec{y}$, written $\vec{x} \succeq \vec{y}$, iff $\forall k \in \{1, \dots, n\}$:

(i) $\displaystyle\sum_{i=1}^{k} x_i^{\downarrow} \geq \sum_{i=1}^{k} y_i^{\downarrow}$,

(ii) equality $k = n$.

**Thm. (Nielsen):**

$$(|\psi\rangle, |\phi\rangle) \in LOCC^{\text{pure, bi}} \iff \vec{\lambda}_\phi \succeq \vec{\lambda}_\psi$$

- Proof relies heavily on Lo & Popescu result.

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# 9. Measures of Entanglement (MoE).

- Want to measure entanglement relative to tasks we do in labs.

<u>Def.</u> A map $S : D_{AB} \to \mathbb{R}$ is a <u>MoE</u> iff

$$(\rho, \sigma) \in LOCC \Rightarrow S(\rho) \geq S(\sigma).$$

<u>Thm. (Vidal):</u>

(i) $\rho, \sigma \in S_{AB} \implies (\rho, \sigma), (\sigma, \rho) \in LOCC,$

(ii) $S$ MoE $\implies \left[ S(|\psi\rangle) \geq S(|\phi\rangle) \iff S(\vec{\lambda}_\psi) \geq S(\vec{\lambda}_\phi) \right].$

Therefore:

(i) $S = \text{Const.}$ on $S_{AB}$.

(ii)

$$(|\psi\rangle, |\phi\rangle) \in LOCC^{\text{Pure,bi}} \overset{\text{Nielsen}}{\iff} \vec{\lambda}_\phi \succeq \vec{\lambda}_\psi$$

$$\overset{S\ \text{MoE}}{\implies} S(|\psi\rangle) \geq S(|\phi\rangle)$$

$$\overset{\text{Vidal}}{\iff} S(\vec{\lambda}_\psi) \geq S(\vec{\lambda}_\phi)$$

- So, $S$ $\text{MoE}^{\text{Pure,bi}}$ iff

$$\vec{\lambda}_\phi \succeq \vec{\lambda}_\psi \implies S(\vec{\lambda}_\psi) \geq S(\vec{\lambda}_\phi).$$

**Def.** A function $G : \mathbb{R}^n_+ \to \mathbb{R}$ is Schur Concave iff

$$\vec{x} \succeq \vec{y} \implies G(\vec{y}) \geq G(\vec{x}).$$

**Cor.** $G$ Schur Concave $\iff$ $G$ $\text{MoE}^{\text{pure, bi}}$.

- $\exists$ many methods for generating Schur Concave fncs.
- In fact, von Neumann and order-$\alpha$ Renyi all follow from such procedures.

**Remark:** $(D_{AB}, \text{LOCC})$ is a Poset:

Most Entangled

$$\begin{tikzcd}
 & \lvert\psi\rangle \arrow[dl] \arrow[dr] & & & \lvert\psi\rangle \arrow[d] \\
\lvert\phi\rangle \arrow[dr] & & \lvert\xi\rangle \arrow[dl] \arrow[r, mapsto, "\text{MoE}"] & & \lvert\xi\rangle \arrow[d] \\
 & \lvert\chi\rangle \arrow[d] & & & \lvert\phi\rangle \arrow[d] \\
 & \lvert\gamma\rangle & & & \lvert\chi\rangle \arrow[d] \\
 & & & & \lvert\gamma\rangle
\end{tikzcd}$$

Least Entangled

- So, every MoE collapses partial order to total order.

**Moral:** There is no "best" MoE.

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