Bell's Theorem

Agenda:

1: Operational QM

2: Bayesian Networks

3: Semialgebraic Statistics

4: Bell's Theorem

5: Tsierison's Problem

6: MIP* = RE

References:

- Causality (Judea Pearl)
- 1802.10119 (Bell)
- 2001.04383 (Ji et al.)

1: Operational QM:

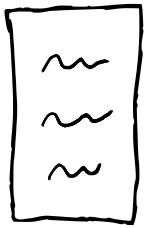
- Quantum foundations gets a bad rap.

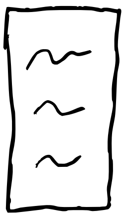
- "It's just philosophy". Does that insinuate that all philosophy is bad?
- Priors.
- Dennet:

  "Scientists sometime deceive themselves into thinking that philosophical ideas are only at best decorations or parasitic commentaries on the hard objective triumphs of science. But there is no such thing as philosophy-free science. There is only science whose philosophical baggage is taken on board without examination."

- QF is, in part, the subfield of QM that teaches us how to think about QM. Such are the interpretations.
- Another side of it gets at the question, what do we really mean by "quantum"? Independent of interpretation!

- View QM as a more general theory of probability based on the theory of Hilbert spaces.

$P =$  $\Longleftrightarrow \rho_P$ (density op.)

$M =$  $\Longleftrightarrow \{M_k\}$ (POVM)

$$\Pr(k \mid P, M) = \mathrm{Tr}(\rho M_k) \quad \text{(Born rule)}$$

- What distr. realizable in this way?

$$\mathcal{D}_{Q,\text{all}} = \{ p \mid p(k) = \mathrm{Tr}(\rho M_k),\ \rho,\ \{M_k\} \}.$$

<u>Classically:</u>

$P \Leftrightarrow X_P$ (random v.)

$M \Leftrightarrow$ Valuations $X_P = k$

$$\Pr(k \mid \rho, M) = \Pr(X = k).$$

$$\mathcal{D}_{C,all} = \{ p \mid p(k) = \Pr(X=k),\ X \}.$$

<u>Q</u>: $\mathcal{D}_{Q,all} \stackrel{?}{=} \mathcal{D}_{C,all}$.

<u>A</u>: Yes. No dist. outside classical prob. theory!

- However, <u>if</u> something like $\mathcal{D}_{Q,all} \neq \mathcal{D}_{C,all}$ were true, <u>then</u> that is surely witnessing "quantum"!

- To get here, I'll give intro. to how mass in QF think about operational QM.

2: Bayesian Networks:

Let $X_1, \dots, X_n$ discrete RVs w/ joint PMF

$$\Pr(X_1, \dots, X_n).$$

Def. Conditional PMF is

$$\Pr(X_i \mid X_1, \dots, X_{i-1}) = \frac{\Pr(X_1, \dots, X_n)}{\Pr(X_1, \dots, X_{i-1})},$$

where

$$\Pr(X_1, \dots, X_{i-1}) = \sum_{X_i = x_i} \cdots \sum_{X_n = x_n} \Pr(X_1, \dots, X_{i-1}, x_i, \dots, x_n).$$

Thm. (Prob. Chain Rule):

$$\Pr(X_1, \dots, X_n) = \prod_i \Pr(X_i \mid X_1, \dots, X_{i-1}).$$

- Proof: recursively apply conditional PMF def.

- It may happen that RV X_i only depends on a subset of $X_1, \dots, X_{i-1}$. That is,

Suppose $\exists\, pa(X_i) \subseteq \{X_1, \dots, X_n\}$ s.t.

$$Pr(X_i \mid X_1, \dots, X_{i-1}) = Pr(X_i \mid pa(X_i)),$$

Graphically:

$$pa(X_i) \longrightarrow X_i$$

i.e. $(|pa(X_i)| = k)$

$$\begin{tikzcd}
pa(X_i)_1 \arrow[dr] & \\
\vdots & X_i \\
pa(X_i)_k \arrow[ur] &
\end{tikzcd}$$

Then,

$$Pr(X_1, \dots, X_n) = \prod_i Pr(X_i \mid pa(X_i)).$$

• Literally defines a graph.

Let $G = (\{X_1, \dots, X_n\}, E)$, where

$$(X_i, X_j) \in E \iff X_j \in Pa(X_i)$$

and

$$Pr_G(X_1, \dots, X_n) := \prod_i Pr(X_i \mid X_j, \text{ where } (X_j, X_i) \in G)$$

$$= \prod_i Pr(X_i \mid Pa(X_i)) \quad .$$

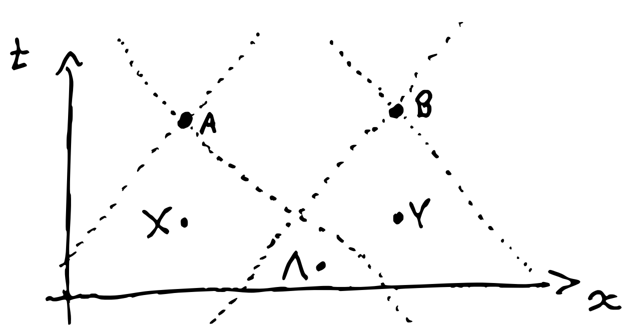
<u>Def.</u> Given $\mathcal{D} \in \mathcal{D}_{all}$, w/ $(X_1, \dots, X_n) \sim \mathcal{D}$, say G is a <u>Bayesian network</u> for $\mathcal{D}$ iff

$$Pr(X_1, \dots, X_n) = Pr_G(X_1, \dots, X_n).$$

Let $\mathcal{D}_G(X_1, \dots, X_n) := \{Pr_G(X_1, \dots, X_n)\}.$

$G=(V,E)$:	$Pr(V)$:
(i) $A \quad\quad B$	$Pr(A)\,Pr(B)$
(ii) $\begin{tikzcd} A & & B \\ & \Lambda \arrow[ul] \arrow[ur] & \end{tikzcd}$	$Pr(A\mid\Lambda)\,Pr(B\mid\Lambda)\,Pr(\Lambda)$
(iii) $\begin{tikzcd} A & & B \\ & \Lambda \arrow[ul] \arrow[ur] & \\ X \arrow[uu] & & Y \arrow[uu] \end{tikzcd}$	$Pr(A\mid X,\Lambda)\,Pr(B\mid Y,\Lambda)\,Pr(X) \cdot Pr(Y)\,Pr(\Lambda)$

Relativistically, (iii) maps to the spacetime diagram:



3. Semialgebraic Statistics:

Given $Pr_G(X_1,\dots,X_n) \in \mathcal{D}_G(X_1,\dots,X_n)$, $\exists$

$$N := |X_1|\cdots|X_n| \text{ Valuations of } (X_1,\dots,X_n).$$

Let $p_i = Pr_G(x_{i1},\dots,x_{in})$ be ith valuation.

Then,

$$p := (p_1,\dots,p_N) \in \left\{ p \in \mathbb{R}^N \;\middle|\; p_i \geq 0,\ \sum_{i=1}^{N} p_i = 1 \right\}$$

Prob. Simplex

Upshot:

$$Pr_G \iff \text{Point in } \mathbb{R}^N$$

$$\mathcal{D}_G = \{Pr_G\} \iff \text{Set in } \mathbb{R}^N$$

Thm: $\mathcal{D}_G \subseteq \mathbb{R}^N$ is an algebraic variety. I.e., $\exists$ $\mathcal{F} \subseteq \mathbb{R}[p_1,\dots,p_N]$ s.t.

notice equals sign.

$$\mathcal{D}_G = \{ p \in \mathbb{R}^N \mid Q(p) = 0,\ Q \in \mathcal{F} \}.$$

Q: Consider

$$\begin{tikzcd} A & & B \\ & \Lambda \arrow[ul] \arrow[ur] & \\ X \arrow[uu] & & Y \arrow[uu] \end{tikzcd}$$

What is $Pr(A, B, X, Y)$ if Λ "hidden"?

A: Marginalize!

$$Pr(A, B, X, Y) = \sum_{\Lambda = \lambda} Pr(A, B, X, Y, \lambda)$$

Thm: Marginalizing X_i in $\mathcal{D}_G(X_1, \dots, X_n)$ makes

$\mathcal{D}_{G, \text{marginal}}$ semi-algebraic set

i.e. $\exists \mathcal{F} \subseteq \mathbb{R}[p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_N]$ s.t.

$$\mathcal{D}_{G, \text{marginal}} = \{ p \in \mathbb{R}^N \mid Q(p) \square 0, \ \square \in \{=, \geq, >\}, \ Q \in \mathcal{F} \}.$$

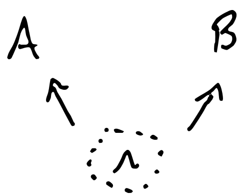
- Tarski-Seidenberg thm.

4. Bell's Theorem:

S_1:

A B

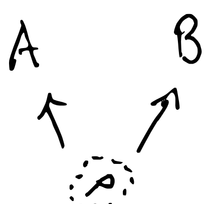
S_2:



Q: $\mathcal{D}_{S_1} \stackrel{?}{=} \mathcal{D}_{S_2}$

A: No! Correlations exist in S_2.

S_3:



$$P_r(A,B) = Tr(M_A \otimes M_B \rho)$$

Q: $\mathcal{D}_{S_2} \stackrel{?}{=} \mathcal{D}_{S_3}$

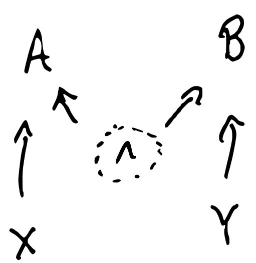
$\underline{A}$: Yes! (Provided Λ "big enough" to simulate correlations.)

- If asked why, give example w/ binary values and simple state $|\psi\rangle$.

"EPR paradox" $\begin{cases} \text{Operationally} & : \quad S_2 = S_3 \\ \text{Metaphysically} & : \quad S_2 \neq S_3 \quad \text{(entanglement)} \end{cases}$

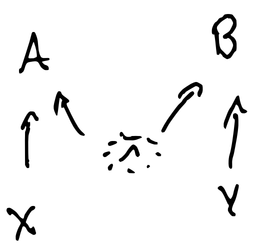
$\underline{\Pr(A, B \mid X, Y)}$

$\underline{S_4}$:



$$\sum_{\Lambda = \lambda} \Pr(A \mid X, \lambda) \Pr(B \mid Y, \lambda) \Pr(\lambda)$$

$\underline{S_5}$:



$$\mathrm{Tr}(M_{A|X} \otimes M_{B|Y}\, \rho)$$

$\underline{Q}$: $\mathcal{D}_{S_4} \stackrel{?}{=} \mathcal{D}_{S_5}$

$\underline{A}$: No! (Bell's Thm.)

<u>WHY?</u> $\exists\, P_r \in \mathcal{D}_{S_S}$ w/ $p = P_r(a,b \mid x,y)$ and inequality $Q \geq 0$ in semi-algebraic set for $\mathcal{D}_{S_E}$ that

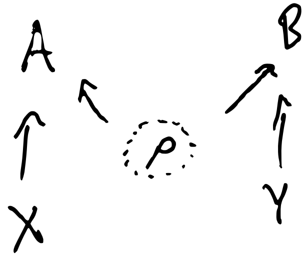
$$Q(p) < 0 .$$ ("Bell inequality").

• Lookup "CHSH inequality".

5: Tsirelson's Problem.

• Gets at how to define (quantumly) separated systems?

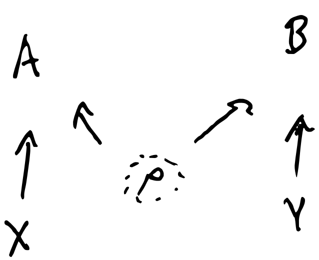
$\underline{S_S}$:



$$\cong \operatorname{Tr}(M_{A|X} \otimes M_{B|Y}\, \rho),$$

where $M_{A|X} \in POVM(\mathcal{H}_A)$, $M_{B|Y} \in POVM(\mathcal{H}_B)$.

$\underline{S_S'}$:



$$\cong \operatorname{Tr}(M_{A|X} M_{B|X}\, \rho),$$

where $M_{A|X}, M_{B|Y} \in POVM(\mathcal{H}_A \otimes \mathcal{H}_B)$ s.t.

$$[M_{A|X}, M_{B|Y}] = 0.$$

- AQFT and Microcausality.

<u>Tsirelson's Problem:</u> $\mathcal{D}_{S_S} \stackrel{?}{=} \mathcal{D}_{S_S'}$?

- Evidently, $\mathcal{D}_{S_S} \subseteq \mathcal{D}_{S_S'}$ but not other way round.

<u>Thm. (Ji et al.):</u> No!

- Open question (and plug for some of my current research), Tsirelson's Problem in other causal structures!

6: $MIP^* = RE$.

- Recall languages are Yes/No problems.

<u>Def.</u> $RE = \{L \mid \exists \text{ TM } M \text{ s.t. } x \in L \Rightarrow M(x) = 1\}$

... If $x \notin L$, then M need not halt!

<u>Note:</u> $HALT \in RE$.

<u>Interactive Proof:</u>

$$\text{Verify} \rightleftarrows \text{Prover}$$

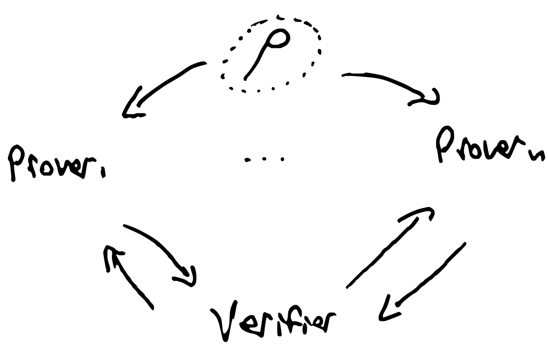
(computationally bounded) (not trustworthy, ∞ power)

- Aside on AI safety.

MIP:

- Prisoner dilemma-like situation.

MIP*: Provers share quantum state



Thm. (Google): $\mathrm{MIP} = \mathrm{NEXP} \subsetneq \mathrm{MIP}^*$.

Thm: (Ji et al.): $\mathrm{MIP}^* = \mathrm{RE}$.

"Entanglement significantly boosts ones comp. power"

- Q: does this hold for other models as well, e.g. for BPP v. BQP?

- Many open questions, talk to me after if interested.

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