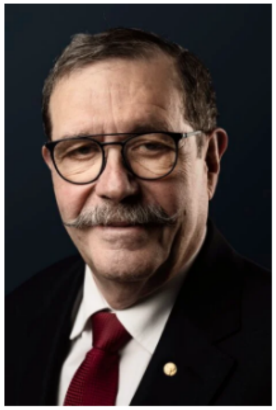
Bell's Theorem and Nonlocal Games

Matthew Fox

Slides available at:

(www.matthewfoxphysics.com/research/talks/018.pdf)

The Nobel Prize in Physics 2022



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Alain Aspect

Prize share: 1/3

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John F. Clauser

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Anton Zeilinger

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The Nobel Prize in Physics 2022 was awarded jointly to Alain Aspect, John F. Clauser and Anton Zeilinger "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science"

(NobelPrize.org)

$\mathsf{MIP}^* = \mathsf{RE}$

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Abstract

We show that the class MIP^* of languages that can be decided by a classical verifier interacting with multiple all-powerful quantum provers sharing entanglement is equal to the class RE of recursively enumerable languages. Our proof builds upon the quantum low-degree test of (Natarajan and Vidick, FOCS 2018) and the classical low-individual degree test of (Ji, et al., 2020) by integrating recent developments from (Natarajan and Wright, FOCS 2019) and combining them with the recursive compression framework of (Fitzsimons et al., STOC 2019).

An immediate byproduct of our result is that there is an efficient reduction from the Halting Problem to the problem of deciding whether a two-player nonlocal game has entangled value 1 or at most $\frac{1}{2}$. Using a known connection, undecidability of the entangled value implies a negative answer to Tsirelson's problem: we show, by providing an explicit example, that the closure C_{qa} of the set of quantum tensor product correlations is strictly included in the set C_{qc} of quantum commuting correlations. Following work of (Fritz, Rev. Math. Phys. 2012) and (Junge et al., J. Math. Phys. 2011) our results provide a refutation of Connes' embedding conjecture from the theory of von Neumann algebras.

Let S be some physical system with measurable properties A, B, and, C.

S	A	B	C
Car	Speed	Color	Position
Star	Mass	Luminosity	Angular Momentum
Electron	x-spin	Y-spin	Z-spin

Measure properties A, B, and C of S many times and collect statistics:

$N_S(A, B, C)$ = # of times S has properties A, B, and C,

$N_S(A, B, \bar{C})$ = # of times S has properties A, B, and NOT C,

$\vdots$

Theorem: $\forall$ systems S with properties A, B, and C, it holds that

$$N_S(A, \bar{B}) + N_S(B, \bar{C}) \geq N_S(A, \bar{C}).$$

"Bell inequality"

Theorem: $N_s(A, \bar{B}) + N_s(B, \bar{C}) \geq N_s(A, \bar{C})$

Proof:

$$\begin{aligned} \text{RHS} &= N_s(A, \bar{C}) \\ &= N_s(A, B, \bar{C}) + N_s(A, \bar{B}, \bar{C}) \end{aligned}$$

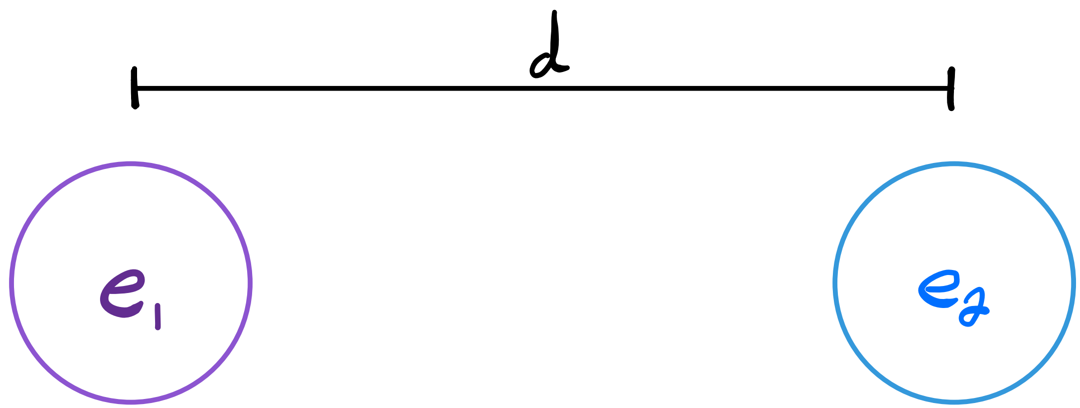
$$\begin{aligned} \text{LHS} &= N_s(A, \bar{B}) + N_s(B, \bar{C}) \\ &= N_s(A, \bar{B}, C) + N_s(A, \bar{B}, \bar{C}) \\ &\quad + N_s(A, B, \bar{C}) + N_s(\bar{A}, B, \bar{C}) \\ &= N_s(A, \bar{B}, C) + N_s(\bar{A}, B, \bar{C}) + \text{RHS}. \end{aligned}$$

Hence,

$$\text{LHS} \geq \text{RHS}. \qquad \blacksquare$$

Quantum Mechanics can violate this inequality!

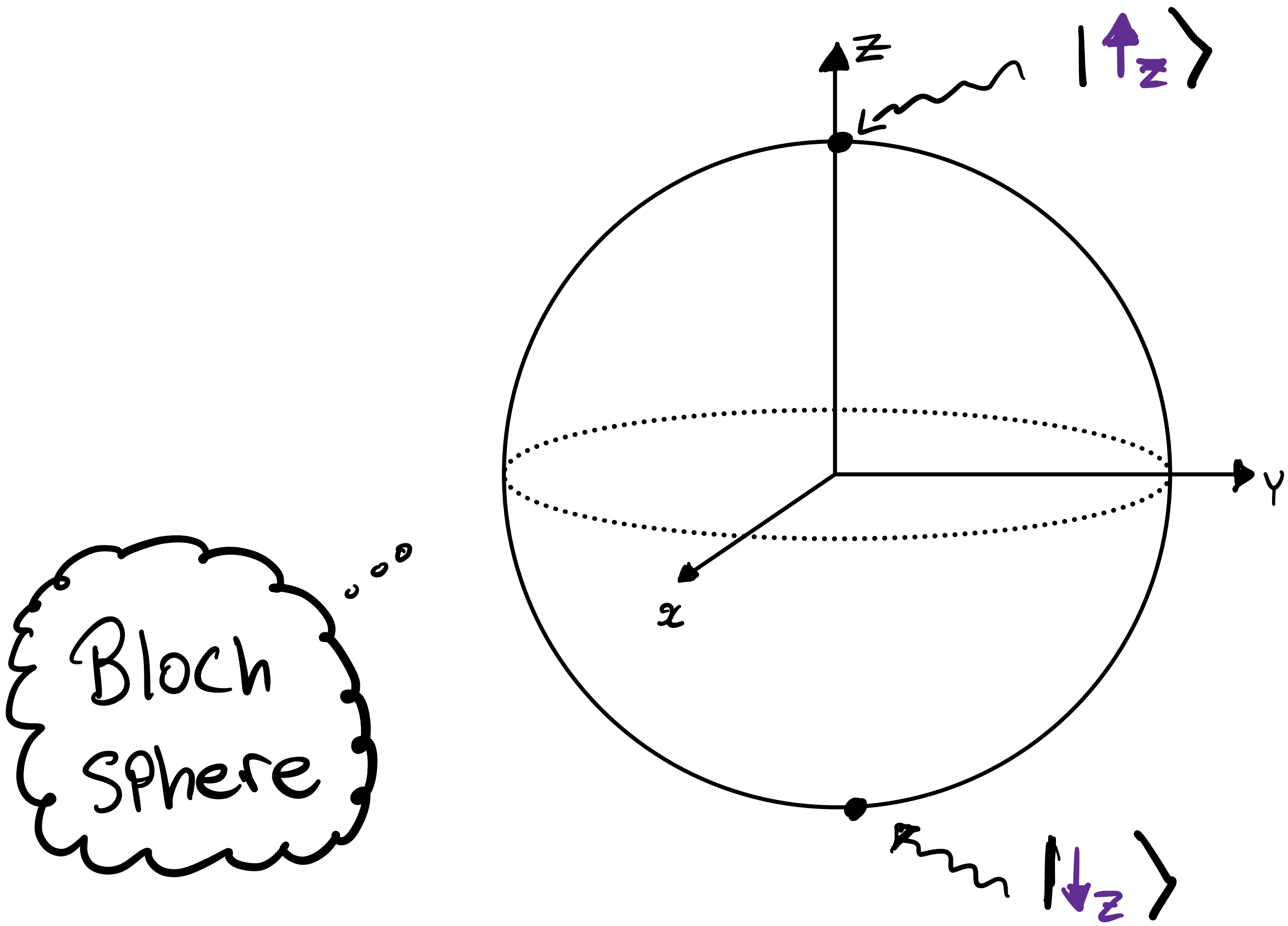
Consider system S consisting of electrons e_1, e_2 such that:

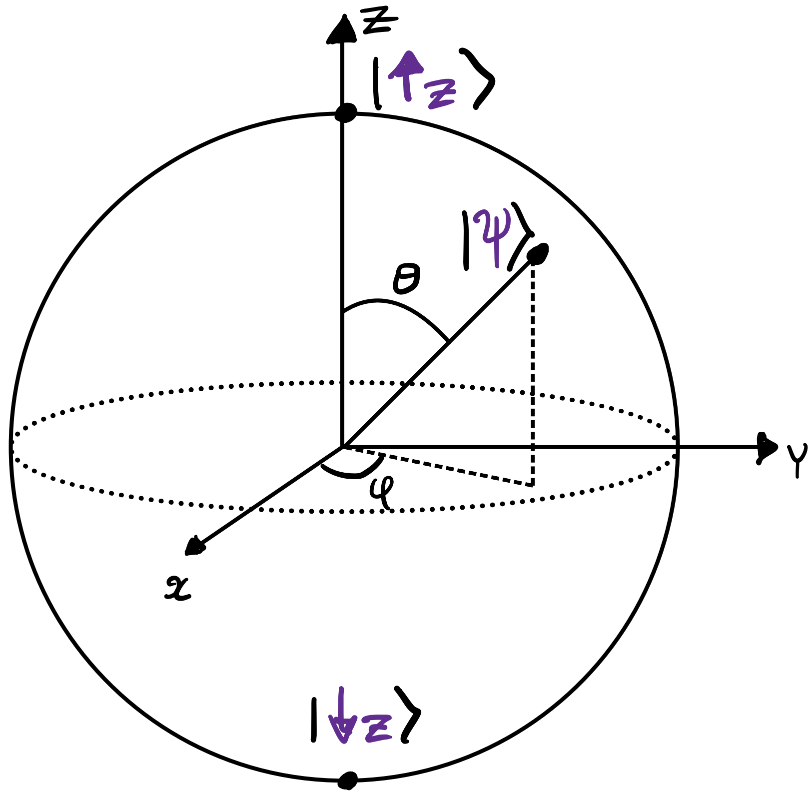


$$\text{State}(e_1, e_2) = \frac{1}{\sqrt{2}}\left(|\uparrow_z \uparrow_z\rangle + |\downarrow_z \downarrow_z\rangle \right)$$

"Maximally entangled"

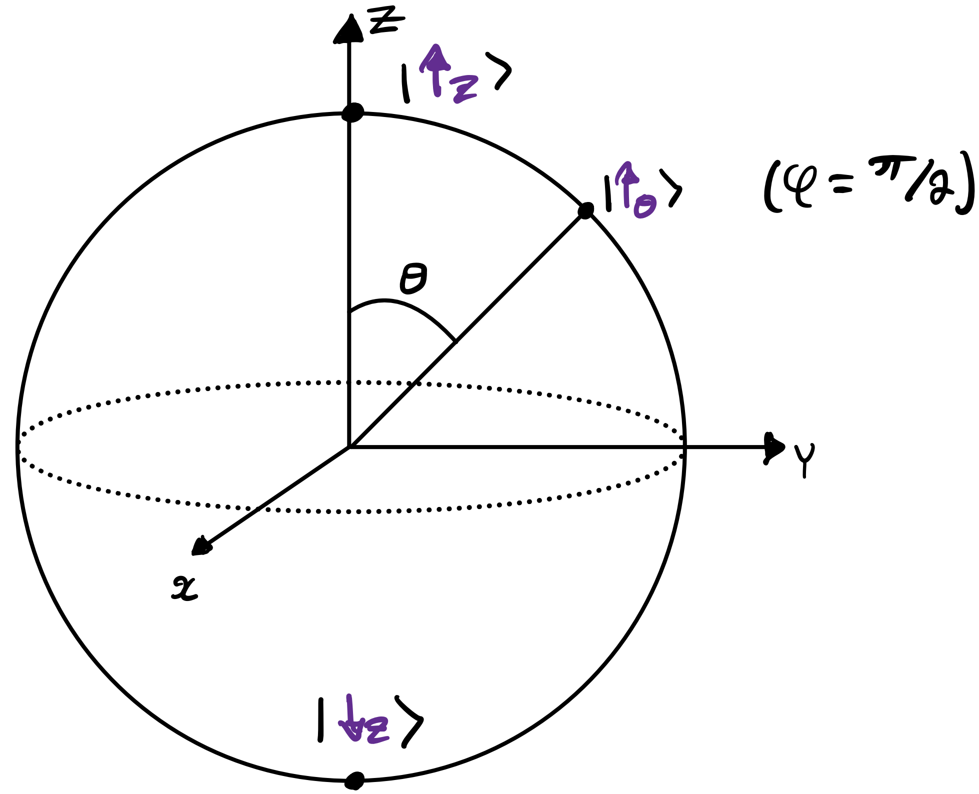
“spin of e_1 along z-axis”





$$|\psi\rangle = \cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\downarrow_z\rangle$$

"Spin of e_1 along θ-axis"



$$|\uparrow_\theta\rangle = \cos\frac{\theta}{2}|\uparrow_z\rangle + \sin\frac{\theta}{2}|\downarrow_z\rangle$$

$$|\downarrow_\theta\rangle = \sin\frac{\theta}{2}|\uparrow_z\rangle - \cos\frac{\theta}{2}|\downarrow_z\rangle$$

Claim: If

S = Maximally entangled electrons e_1, e_2

A = spin of e_1 along z-axis

B = spin of e_2 along θ-axis

C = spin of e_2 along 2θ-axis

and

$$N_S(A, \bar{B}) + N_S(B, \bar{C}) \geq N_S(A, \bar{C}),$$

then

$$\frac{1}{2} \geq 1.$$

Conclude: QM Violates the Bell inequality!

Run experiment M times.

Then:

$$N_s(A, \bar{B}) = M \cdot P_r[A, \bar{B}]$$

$$= M \cdot P_r[A \mid \bar{B}]\, P_r[\bar{B}].$$

But

B = spin of e_2 along θ-axis

$$P_r[\bar{B}] = P_r[\text{state}(e_2) \neq |\uparrow_\theta\rangle]$$

$$= 1 - P_r[\text{state}(e_2) = |\uparrow_\theta\rangle].$$

Recall,

$$\text{State}(e_1, e_2) = \frac{1}{\sqrt{2}}\left(|\uparrow_z \uparrow_z\rangle + |\downarrow_z \downarrow_z\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow_\theta \uparrow_\theta\rangle + |\downarrow_\theta \downarrow_\theta\rangle\right)$$

Therefore,

$$\Pr\left[\text{State}(e_2) = |\uparrow_\theta\rangle\right] = \Pr\left[\text{State}(e_1, e_2) = |\uparrow_\theta \uparrow_\theta\rangle\right]$$

$$= \frac{1}{2}.$$

$$\Rightarrow \quad \Pr\left[\bar{B}\right] = \frac{1}{2}$$

Then:

$$N_s(A, \bar{B}) = M \cdot Pr[A \mid \bar{B}] \, Pr[\bar{B}]$$
$$= \frac{M}{2} Pr[A \mid \bar{B}]$$

Recall:

A = spin of e_1 along z-axis

So:

$$Pr[A \mid \bar{B}] = Pr\left[\text{state}(e_1) = |\uparrow_z\rangle \mid \text{state}(e_2) = |\downarrow_\theta\rangle\right]$$
$$= \sin^2\frac{\theta}{2}$$

$$|\downarrow_\theta\rangle = \sin\frac{\theta}{2}|\uparrow_z\rangle - \cos\frac{\theta}{2}|\downarrow_z\rangle$$

S = Maximally entangled electrons e_1, e_2

A = spin of e_1 along z-axis

B = spin of e_2 along θ-axis

C = spin of e_2 along 2θ-axis

$$\underbrace{N_S(A, \bar{B})}_{\frac{M}{2}\sin^2\frac{\theta}{2}} + \underbrace{N_S(B, \bar{C})}_{\frac{M}{2}\sin^2\frac{\theta}{2}} \geq \underbrace{N_S(A, \bar{C})}_{\frac{M}{2}\sin^2\theta} .$$

Left with the inequality

$$\frac{M}{2}\sin^2\frac{\theta}{2} + \frac{M}{2}\sin^2\frac{\theta}{2} \geq \frac{M}{2}\sin^2\theta$$

$$\Longleftrightarrow$$

$$2\sin^2\frac{\theta}{2} \geq \sin^2\theta$$

<u>Q:</u> Is this true for all θ?

A: No! Take $0 < \theta \lll 1$ so that

$$\sin\theta \simeq \theta.$$

Then,

$$2\sin^2\frac{\theta}{2} \geq \sin^2\theta$$

$$\Longleftrightarrow$$

$$\frac{\theta^2}{2} \geq \theta^2$$

$$\Longleftrightarrow$$

$$\frac{1}{2} \geq 1$$

Contradiction!

(A Version of) Bell's Theorem:

If

S = Maximally entangled electrons e_1, e_2

A = spin of e_1 along z-axis

B = spin of e_2 along θ-axis $(0<\theta\ll 1)$

C = spin of e_2 along 2θ-axis

then

$$N_S(A,\bar{B}) + N_S(B,\bar{C}) < N_S(A,\bar{C}).$$

"Violation of the Bell inequality"

Let's go back...

Theorem: $\forall$ systems S <u>with</u> properties A, B, and C, it holds that

$$N_S(A, \overline{B}) + N_S(B, \overline{C}) \geq N_S(A, \overline{C}).$$

Bell's Theorem:

If

S = Maximally entangled electrons e_1, e_2

A = spin of e_1 along z-axis

B = spin of e_2 along θ-axis $(0 < \theta \ll 1)$

C = spin of e_2 along 2θ-axis

then

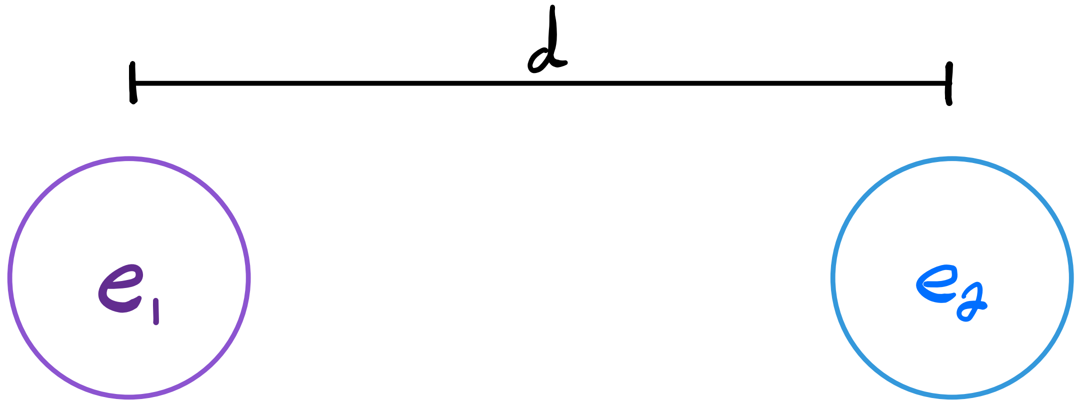
$$N_S(A, \overline{B}) + N_S(B, \overline{C}) < N_S(A, \overline{C}).$$

"Violation of the Bell inequality"

Corollary:

- Maximally entangled electrons cannot have definite z, θ, and 2θ-spin simultaneously.

- I.e., there are no "local hidden variables".



$$\text{State}(e_1, e_2) = \frac{1}{\sqrt{2}}\left(|\uparrow_z \uparrow_z\rangle + |\downarrow_z \downarrow_z\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow_\theta \uparrow_\theta\rangle + |\downarrow_\theta \downarrow_\theta\rangle\right)$$

Independent of $d \Rightarrow$ Nonlocality.

<u>Q</u>: Is lesson of Bell's theorem that QM is nonlocal?

<u>A</u>: Maybe...

<u>Hidden Assumption</u>: Measurements have one outcome.

<u>Q</u>: Is that reasonable?

Everettian (Many Worlds) QM

- Take seriously that everything is quantum.
- Measure electron e spin with detector D and human H in the room:

$$\text{State}_{\text{initial}}(e, D, H) = \left(\alpha|\uparrow\rangle + \beta|\downarrow\rangle\right)|\text{ready}\rangle|\text{ready}\rangle$$

$$\text{State}_{\text{final}}(e, D, H) = \alpha|\uparrow\rangle|D \text{ says "}\uparrow\text{"}\rangle|H \text{ sees "}\uparrow\text{"}\rangle + \beta|\downarrow\rangle|D \text{ says "}\downarrow\text{"}\rangle|H \text{ sees "}\downarrow\text{"}\rangle$$

"Each branch a world"

Clauser, Horne,
Shimony, Holt

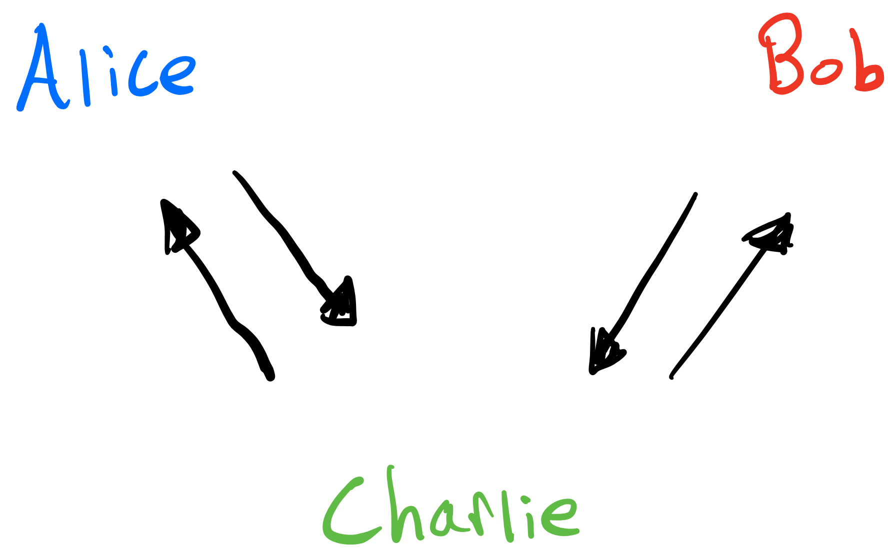
The CHSH inequality

and

Nonlocal Games

CHSH and Nonlocal Games

- Exploit Bell inequality for computational speedup.



<u>Note:</u> No communication between Alice and Bob!

Stage 0 (before game begins):

- Alice, Bob — in collaboration — choose Boolean random variables

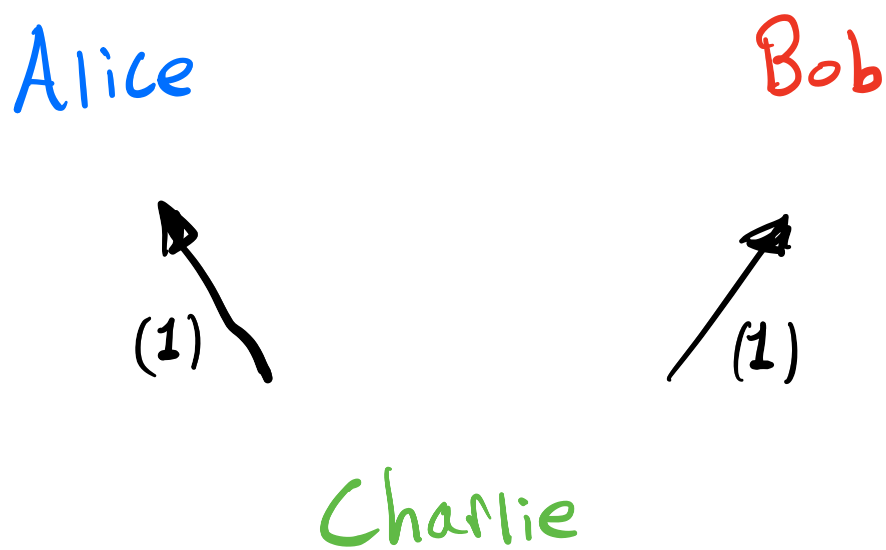
$$a : \{0,1\} \longrightarrow \{0,1\}$$

$$b : \{0,1\} \longrightarrow \{0,1\}.$$

- Their <u>strategy</u>.
- Denote $a_x = a(x)$, $b_y = b(y)$.

Stage 1:

- Charlie independently and uniformly selects $x, y \in \{0,1\}$.
- Sends x to Alice, y to Bob



Stage 2:

Using their strategy,

- Alice chooses $a_x \in \{0,1\}$
- Bob chooses $b_y \in \{0,1\}$

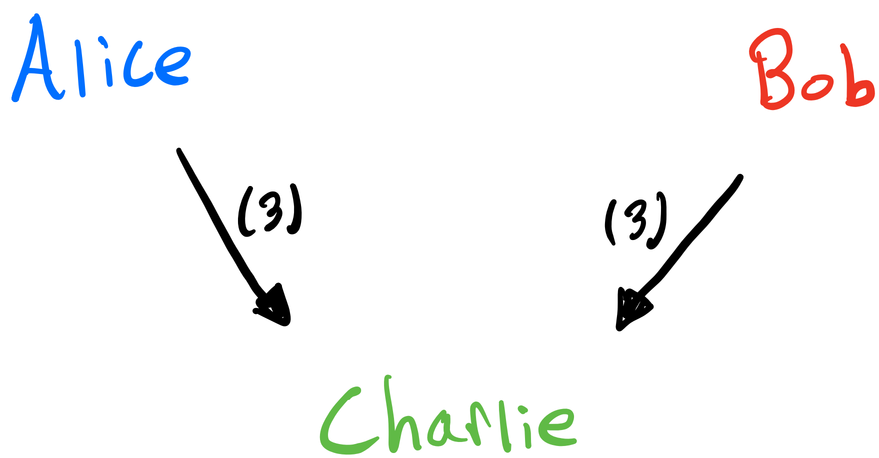
(2) Alice

Bob (2)

Charlie

Stage 3:

- Alice sends Charlie a_x
- Bob sends Charlie b_y



Stage 4:

- Charlie computes

$$a_x \oplus b_y = a_x + b_y \pmod 2$$

- Charlie declares
  - "win" if $a_x \oplus b_y = xy$
  - "lose" otherwise.

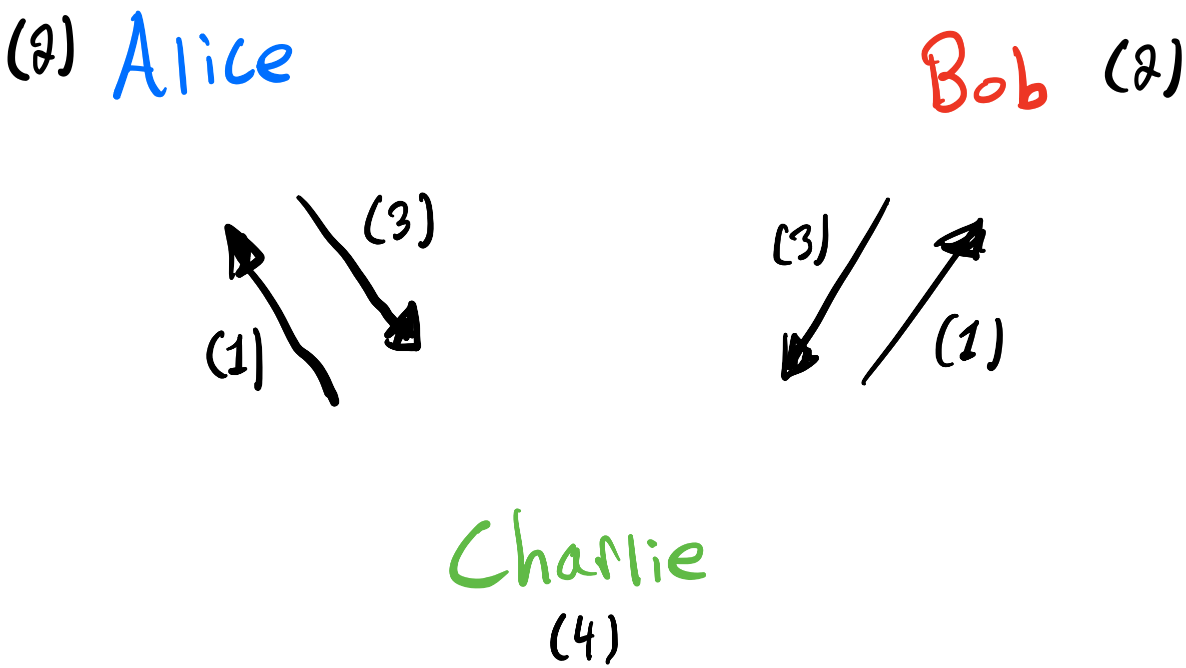
Alice   Bob

Charlie

(4)

The Nonlocal XOR Game

(a.k.a. CHSH Game)



Winning Condition: $a_x \oplus b_y = xy$.

Q: What strategy maximizes

$$P_{win} = \Pr[a_x \oplus b_y = xy] \ ?$$

Winning Table:

x	y	Winning Condition
0	0	$a_0 \oplus b_0 = 0$
0	1	$a_0 \oplus b_1 = 0$
1	0	$a_1 \oplus b_0 = 0$
1	1	$a_1 \oplus b_1 = 1$

One Strategy:
- $a \equiv 1,\ b \equiv 1 \implies P_{win} = 3/4$.
- Can they do better?

Claim: Optimal $P_{win} < 1$.

Proof: Suppose not. Then $\exists$ functions

$$a : \{0,1\} \to \{0,1\}$$
$$b : \{0,1\} \to \{0,1\}$$

such that:

$$a_0 \oplus b_0 = 0$$
$$a_0 \oplus b_1 = 0$$
$$a_1 \oplus b_0 = 0$$
$$a_1 \oplus b_1 = 1$$

But then:

$$1 = 0 \oplus 0 \oplus 0 \oplus 1$$
$$= \sum_{x,y} a_x \oplus b_y = 2(a_0 \oplus a_1 \oplus b_0 \oplus b_1)$$
$$= 0.$$

Contradiction!

Conclusion: optimal strategy necessarily probabalistic.

Define $A_x = (-1)^{a_x}$, $B_y = (-1)^{b_y}$ $\in \{-1, 1\}$

x	y	Winning Condition	$P_{win \mid xy}$	$\mathbb{E}[A_x B_y]$
0	0	$A_0 B_0 = 1$	$P_{win \mid 00}$	$P_{win \mid 00} + (-1)(1 - P_{win \mid 00})$ $= 2P_{win \mid 00} - 1$
0	1	$A_0 B_1 = 1$	$P_{win \mid 01}$	$2P_{win \mid 01} - 1$
1	0	$A_1 B_0 = 1$	$P_{win \mid 10}$	$2P_{win \mid 10} - 1$
1	1	$A_1 B_1 = -1$	$P_{win \mid 11}$	$1 - 2P_{win \mid 11}$

Define

$$C = A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1$$

Claim: $\mathbb{E}[C] = 8P_{win} - 4.$

Proof: $\mathbb{E}[C] = 2\left(P_{win|00} + P_{win|01} + P_{win|10} + P_{win|11}\right) - 4.$

Since

$$Pr(x=0) = Pr(x=1) = Pr(y=0) = Pr(y=1) = \frac{1}{2},$$

by law of total probability,

$$P_{win} = \sum_{x,y} P_{win|xy} Pr(x,y)$$

$$= \sum_{x,y} P_{win|xy} Pr(x) Pr(y)$$

$$= \frac{1}{4} \sum_{x,y} P_{win|xy} \Rightarrow \mathbb{E}[C] = 8P_{win} - 4. \quad \blacksquare$$

Claim: $|\mathbb{E}[C]| \leq 2$ ... CHSH inequality

Proof: Notice

- $A_0 = A_1 \implies A_0 + A_1 = \pm 2, \ A_0 - A_1 = 0$
- $A_0 \neq A_1 \implies A_0 + A_1 = 0, \ A_0 - A_1 = \pm 2$
- $C = A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1$

$$= (A_0 + A_1) B_0 + (A_0 - A_1) B_1$$

Therefore,

$$C = \pm 2 \implies |C| = 2$$

and so

$$|\mathbb{E}[C]| \leq \mathbb{E}[|C|] = 2. \qquad \blacksquare$$

Altogether,

- $\mathbb{E}[c] = 8p_{win} - 4$
- $|\mathbb{E}[c]| \leq 2$.

Among "good" strategies for which $p_{win} \geq \frac{1}{2}$,

$$\mathbb{E}[c] = |\mathbb{E}[c]|.$$

Therefore, for any "good" strategy,

$$8p_{win} - 4 \leq 2$$

$$\Longleftrightarrow$$

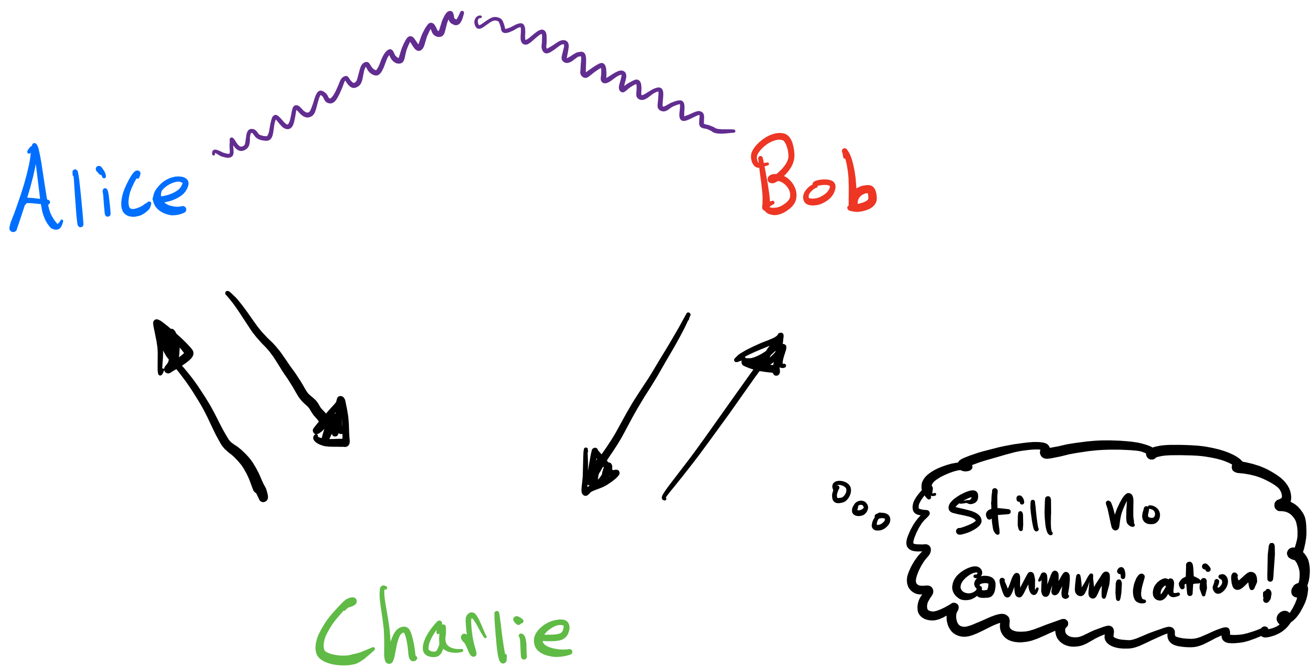
$$\boxed{p_{win} \leq 3/4 = 75\%}$$

... "optimal classical strategy"

Q: Can Alice and Bob do better if they use a quantum strategy?

Setup:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$



Stage 0 (before game begins):

- Alice, Bob — in collaboration — choose measurement operators
  - $M_A(x)$
  - $M_B(y)$

  and measure their side of $|\Psi^-\rangle$.

- Their Quantum Strategy.

Claim: $\exists$ quantum strategy such that

$P_{win} \simeq 85\%$.

Proof:

Warning: Some Serious Math ahead!

Def/Claim:

The Pauli Matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and satisfy, for all $i, j \in \{1,2,3\}$,

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{I} + i \sum_k \varepsilon_{ijk} \sigma_k$$

Kronecker delta

2x2 identity matrix

Levi-Civita symbol

(For Proof, See Nielsen and Chuang)

Claim: Let σ be any Pauli matrix and write

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B\right).$$

Then,

$$\sigma_A |\Psi^-\rangle = -\sigma_B |\Psi^-\rangle.$$

σ_A means "act with σ on system A"

(For proof, see Nielsen and Chuang)

Let

- $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, "vector of matrices"
- $\hat{R} = (R_1, R_2, R_3) \in \mathbb{R}^3$ (unit vector)
- $\hat{S} = (S_1, S_2, S_3) \in \mathbb{R}^3$ (unit vector)

Then

$$A(\hat{R}) = \vec{\sigma}_A \cdot \hat{R} = R_1 \sigma_{A1} + R_2 \sigma_{A2} + R_3 \sigma_{A3}$$
$$= \sum_i R_i \sigma_{Ai}$$

$$B(\hat{S}) = \vec{\sigma}_B \cdot \hat{S} = S_1 \sigma_{B1} + S_2 \sigma_{B2} + S_3 \sigma_{B3}$$
$$= \sum_i S_i \sigma_{Bi}$$

- The vectors $\hat{R}$ and $\hat{S}$ determine the quantum strategy (measurement operators).

<u>Claim:</u> $\mathbb{E}\left[A(\hat{R})B(\hat{S})\right] = -\hat{R}\cdot\hat{S}.$

<u>Proof:</u> By definition (in QM):

$$\mathbb{E}\left[A(\hat{R})B(\hat{S})\right] = \langle\Psi^-|\,A(\hat{R})B(\hat{S})\,|\Psi^-\rangle$$
$$= \langle\Psi^-|(\vec{\sigma}_A\cdot\hat{R})(\vec{\sigma}_B\cdot\hat{S})|\Psi^-\rangle.$$

But for any σ,

$$\sigma_A|\Psi^-\rangle = -\sigma_B|\Psi^-\rangle.$$

From before

Therefore,

$$\mathbb{E}\left[A(\hat{R})B(\hat{S})\right] = -\langle\Psi^-|(\vec{\sigma}_A\cdot\hat{R})(\vec{\sigma}_A\cdot\hat{S})|\Psi^-\rangle.$$
$$= -\sum_{i,j}R_iS_j\,\langle\Psi^-|\sigma_{A_i}\sigma_{A_j}|\Psi^-\rangle.$$

Claim: $\mathbb{E}\left[A(\hat{R})B(\hat{S})\right] = -\hat{R}\cdot\hat{S}$.

Proof (Cont.): But recall

$$\sigma_i \sigma_j = \delta_{ij}\mathbb{I} + i\sum_k \varepsilon_{ijk}\sigma_k .$$

Therefore,

$$\mathbb{E}\left[A(\hat{R})B(\hat{S})\right] = -\sum_{i,j} R_i S_j \langle \Psi^- | \sigma_{Ai}\sigma_{Aj} | \Psi^- \rangle$$

$$= -\sum_{i,j} R_i S_j \langle \Psi^- | \left(\delta_{ij}\mathbb{I} + i\sum_k \varepsilon_{ijk}\sigma_{Ak}\right) | \Psi^- \rangle$$

$$= -\sum_i R_i S_i \langle \Psi^- | \Psi^- \rangle$$

$$= -\hat{R}\cdot\hat{S}.$$

$\blacksquare$

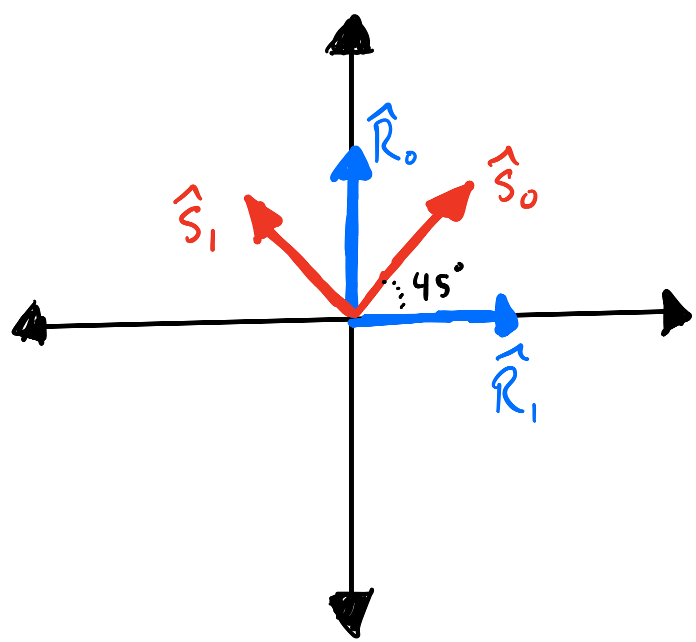
Alice and Bob's Quantum Strategy

Alice:

- If $x=0$,
  → measure
  $A_0 = \vec{\sigma}_A \cdot \hat{R}_0$
- If $x=1$,
  → measure
  $A_1 = \vec{\sigma}_A \cdot \hat{R}_1$

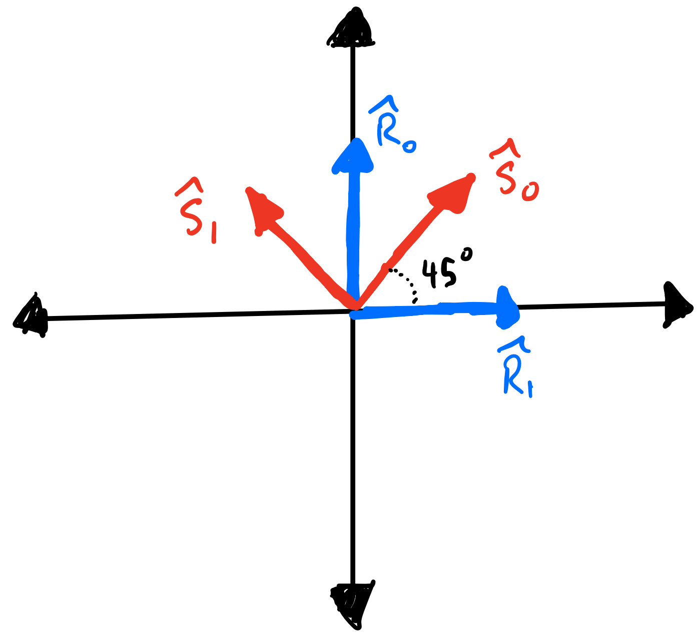
Bob:

- If $y=0$,
  → measure
  $B_0 = \vec{\sigma}_B \cdot \hat{S}_0$
- If $y=1$,
  → measure
  $B_1 = \vec{\sigma}_B \cdot \hat{S}_1$



Then,

- $\mathbb{E}[A_0 B_0] = -\hat{R}_0 \cdot \hat{S}_0$
  $= -\frac{1}{\sqrt{2}}$
- $\mathbb{E}[A_0 B_1] = -\frac{1}{\sqrt{2}}$
- $\mathbb{E}[A_1 B_0] = -\frac{1}{\sqrt{2}}$
- $\mathbb{E}[A_1 B_1] = +\frac{1}{\sqrt{2}}$



Recall: $C = A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1$.

So, with this strategy,

$$\mathbb{E}[C] = -2\sqrt{2} \rightarrow |\mathbb{E}[C]| = 2\sqrt{2}.$$

We just proved $\exists$ quantum strategy such that

$$|\mathbb{E}[c]| = 2\sqrt{2}.$$

But earlier we proved the CHSH inequality:

$$|\mathbb{E}[c]| \leq 2.$$

$$\boxed{\text{Quantum Mechanics Violates the CHSH inequality!}}$$

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Earlier we also proved

$$\mathbb{E}[C] = 8P_{win} - 4.$$

Among "good" strategies for which $P_{win} \geq \frac{1}{2}$,

$$\mathbb{E}[C] = |\mathbb{E}[C]|.$$

Therefore, our quantum strategy gives

$$8P_{win} - 4 = 2\sqrt{2}$$

$$\rightarrow \boxed{P_{win} = \frac{1}{2} + \frac{1}{2\sqrt{2}} \simeq 85\%}$$

<u>Conclude:</u> QM <u>provably</u> outperforms CM in XOR nonlocal game!

Thank You!

Slides