

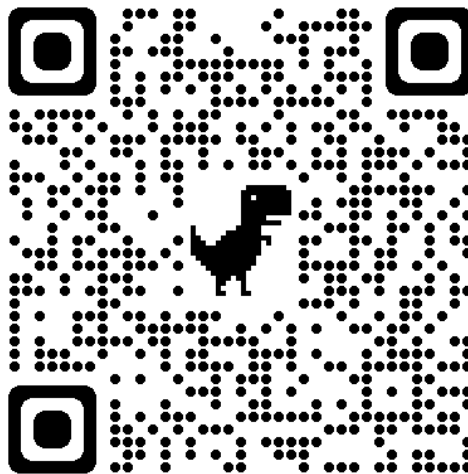
Abstract Complexity

&

The Blum Axioms

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Slides Available At:



Recall:

A function $\varphi: X \rightarrow Y$ is:

- **Partial** iff $\text{dom}(\varphi) \neq X$
- **total** iff $\text{dom}(\varphi) = X$
- **Partial Computable (P.C.)** iff $\exists$ a (two-way, single-tape, deterministic) TM M s.t.

φ_M is the
"proper function"
of M

$$(i) \text{ dom}(\varphi_M) = \text{dom}(\varphi)$$

$$(ii) \forall x \in \text{dom}(\varphi): \varphi_M(x) = \varphi(x)$$

Corollary: $\{\text{p.c. functions } \varphi\} \cong_{\text{set}} \mathbb{N}$

i.e.

$\varphi_1, \varphi_2, \varphi_3, \dots$

Some φ_i are "harder" to compute because they require:

- more time on a DTM
- more space on an NTM
- more tape-reversals on a DTM
- more "ink" on a DTM
- more logic gates in a uniform circuit
- etc.

Q: What, Mathematically, is a complexity measure
on the space of p.c. functions $\varphi_1, \varphi_2, \varphi_3, \dots$?

Def. [Blum '67]:

An **abstract Complexity Measure** Φ is a p.c. function such that:

(i) $\forall x: \Phi(i, x) \downarrow$ iff $\varphi_i(x) \downarrow$,

(ii) $\{(i, x, m) \mid \Phi(i, x) = m\}$ is decidable.

Ex: • Φ = deterministic time/space (**dtime/dspace**)

• Φ = non-deterministic space

Non Ex:

• Φ = # of random coins used by a PTM.

Def. [Blum, '67]:

Together, Φ and a total p.c. function $f: \mathbb{N} \rightarrow \mathbb{N}$ define a complexity class:

$$C_f^\Phi := \{ \varphi_i \mid \Phi(i, n) \leq f(n) \text{ a.e.} \}.$$

“almost everywhere”

Ex: • $C_{O(n^4)}^{\text{dtime}} = \text{FDTIME}(O(n^4))$

• $C_{2^{O(n)}}^{\text{dspace}} = \text{FEXPSPACE}$

Gap Theorem [Borodin, '72; Trakhtenbrot, '67].

$\forall \Phi, \forall \text{ total p.c. } f(n) \geq n, \exists \text{ total p.c. } t \text{ s.t.}$

$$C_{t}^{\Phi} = C_{f \circ t}^{\Phi}$$

i.e.

$$\{ \varphi_i \mid \Phi(i, n) \leq t(n) \text{ a.e.} \}$$

=

$$\{ \varphi_i \mid \Phi(i, n) \leq (f \circ t)(n) \text{ a.e.} \}.$$

Corollary:

$\Phi = \text{dtime}$. Then, $\forall$ total p.c. $f(n) \geq n$, $\exists$ total p.c.

t s.t.

$$\text{DTIME}(t(n)) = \text{DTIME}(f(t(n))).$$

Moral: "There are arbitrarily large gaps in the DTIME complexity hierarchy."

Quiz: Does this violate the time-hierarchy theorem?

$$\text{DTIME}(o(t(n))) \neq \text{DTIME}(t(n) \log t(n))?$$

Speedup Theorem [Blum, '71]:

$\forall \Phi, \forall \text{ total p.c. } f, \exists \text{ total p.c. } g \text{ s.t. } \forall \varphi_i = g,$

$\exists \varphi_i = g \text{ satisfying:}$

$$f(n, \Phi(j, n)) < \Phi(i, n) \text{ a.e.}$$

Corollary 3

$\Phi = \text{dtime}$. Then, $\forall$ total p.c. f , $\exists$ a language L s.t. $\forall$ DTMs M_i that decide L , $\exists M_j$ that decides L and that satisfies:

$$f(\text{dtime}(i, x)) < \text{dtime}(j, x) \text{ a.e.}$$

Moral: " $\exists$ computable functions with no asymptotically fastest algorithm because they can be endlessly sped up."

Union Theorem [McCreight-Meyer, '69]:

Let $f_1, f_2, f_3, \dots$ be a computable list of total p.c. functions s.t. $\forall i, n, f_i(n) \leq f_{i+1}(n)$. Then, $\forall \Phi, \exists$ total p.c. f s.t.

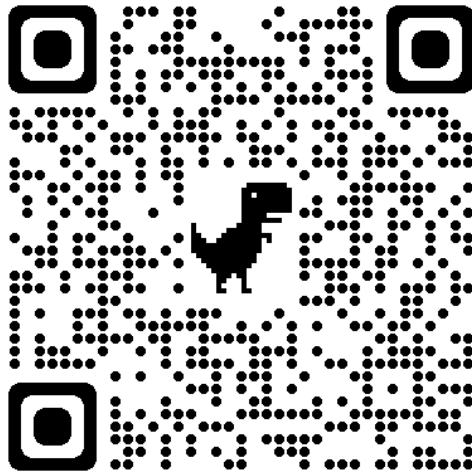
$$C_f^\Phi = \bigcup_i C_{f_i}^\Phi.$$

Corollary/Moral:

$\exists$ total p.c. f_1 , f_2 , and f_3 s.t.:

- $P = \text{DTIME}(f_1)$
- $NP = \text{NTIME}(f_2)$
- $PSPACE = \text{SPACE}(f_3)$

Thank You!



slides

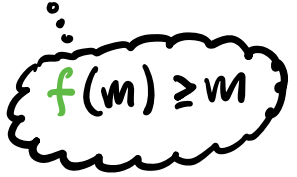


Gap Theorem ($\Phi = \text{dtime}$):

$\forall$ total p.c. $f(n) \geq n$, $\exists$ total p.c. t s.t.

$$\text{DTIME}(t(n)) = \text{DTIME}(f(t(n))).$$

Proof:

- Enumerate DTMs $M_1, M_2, M_3, \dots$, with $T_i(x) = \text{dtime}(i, x)$.
 - $\forall n$, define
$$t(n) := \min \{m \mid \forall (i \leq n) (T_i(n) \leq f(m) \Rightarrow T_i(n) \leq m)\}.$$
 - $t(n)$:
 $m = 0$
 for $i \in [n]$:
 if $m < T_i(n) \leq f(m)$:
 $m = T_i(n)$
 return m .

 - If M_i runs in time $f(t(n))$, then $T_i(n) \leq t(n)$.
- So,
- $$\text{DTIME}(t(n)) = \text{DTIME}(f(t(n))).$$

QED

Speedup Theorem ($\Phi = \text{dtime}$):

$\forall$ total p.c. f , $\exists$ a language L s.t. $\forall$ DTM's M_i
that decide L , $\exists M_j$ that decides L and satisfies:

$$f(\text{dtime}(j, x)) < \text{dtime}(i, x) \text{ a.e.}$$

Proof Idea:

- Let $f^n = f \circ f \circ \dots \circ f$ n times.

- Construct (diagonalization) $L \in \{0\}^*$ s.t.

(i) $\forall M_i$ accepting L ,

$$\text{dtime}(i, 0^n) > f^{n-i}(2) \text{ a.e.}$$

(ii) $\forall k, \exists M_k$ accepting L s.t.

$$\text{dtime}(k, 0^n) \leq f^{n-k}(2) \text{ a.e.}$$

- Then,

$$(ii) \Rightarrow \exists M_k \text{ accepting } L \text{ s.t. } \text{dtime}(k, 0^n) \leq f^{n-k-1}(2) \text{ a.e.}$$

So

$$(i) \Rightarrow f(\text{dtime}(i, 0^n)) \leq f^{n-i}(2) < \text{dtime}(i, 0^n).$$

Union Theorem:

Let $f_1, f_2, f_3, \dots$ be a computable list of total p.c. functions s.t. $\forall i, n, f_i(n) \leq f_{i+1}(n)$. Then, $\forall \Phi, \exists$ total p.c. f s.t.

$$\mathcal{C}_f^\Phi = \bigcup_i \mathcal{C}_{f_i}^\Phi,$$

where

$$\mathcal{C}_f^\Phi := \{ \varphi_i \mid \Phi(i, n) \leq f(n) \text{ a.e.} \}.$$

Proof Idea:

- Construct f (diagonalization) satisfying:

$$(i) \forall k \quad f(n) \geq f_k(n) \quad \text{a.e.}$$

(ii) $\forall i$, if

$$(\forall k: \Phi(i, n) > f_k(n) \text{ i.o.}),$$

then

$$f(n) < \Phi(i, n) \text{ i.o.}$$

"infinitely often"

- Then,

$$(i) \Rightarrow C_{f_i}^{\Phi} \subseteq C_f^{\Phi}$$

$$(ii) \Rightarrow \text{If } \Phi(i, n) \leq f(n) \text{ a.e., then } \exists f_i \text{ s.t.}$$
$$\Phi(i, n) \leq f_i(n).$$

$$\Rightarrow C_f^{\Phi} \subseteq \bigcup_i C_{f_i}^{\Phi}.$$