Abstract Complexity & The Blum Axioms

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Slides Available At:

Recall:

A function $\varphi : X \to Y$ is:

- **Partial** iff $\mathrm{dom}(\varphi) \neq X$
- **total** iff $\mathrm{dom}(\varphi) = X$
- **Partial Computable (P.C.)** iff $\exists$ a (two-way, single-tape, deterministic) TM M s.t.

  (i) $\mathrm{dom}(\varphi_M) = \mathrm{dom}(\varphi)$

  (ii) $\forall x \in \mathrm{dom}(\varphi) : \varphi_M(x) = \varphi(x)$

φ_M is the "proper function" of M

Corollary: $\{\text{p.c. functions } \varphi\} \cong_{set} \mathbb{N}$

i.e.

$\varphi_1, \varphi_2, \varphi_3, \ldots$

Some φ_i are "harder" to compute because they require:

- more time on a DTM
- more space on an NTM
- more tape-reversals on a DTM
- more "ink" on a DTM
- more logic gates in a uniform circuit
- etc.

Q: What, mathematically, *is* a complexity measure on the space of p.c. functions $\varphi_1, \varphi_2, \varphi_3, \ldots$?

Def. [Blum '67]:

An **abstract complexity measure** Φ is a p.c. function such that:

(i) $\forall x : \Phi(i,x)\downarrow$ iff $\varphi_i(x)\downarrow$,

(ii) $\{(i,x,m) \mid \Phi(i,x) = m\}$ is decidable.

Ex:

- Φ = deterministic time/space (dtime/dspace)
- Φ = non-deterministic space

Non Ex:

- Φ = # of random coins used by a PTM.

Def. [Blum, '67]:

Together, Φ and a total p.c. function $f: \mathbb{N} \to \mathbb{N}$ define a complexity class:

$$C_f^{\Phi} := \{ \varphi_i \mid \Phi(i,n) \leq f(n) \text{ a.e.} \}.$$

"almost everywhere"

Ex:

- $C^{\text{dtime}}_{O(n^{48})} = \text{FDTIME}(O(n^{48}))$
- $C^{\text{dspace}}_{2^{n^{O(1)}}} = \text{FEXPSPACE}$

Gap Theorem [Borodin, '72; Trakhtenbrot, '67].

$\forall \Phi$, $\forall$ total p.c. $f(n) \geq n$, $\exists$ total p.c. t s.t.

$$C_t^{\Phi} = C_{f \circ t}^{\Phi}$$

i.e.

$$\{ \varphi_i \mid \Phi(i,n) \leq t(n) \text{ a.e.} \}$$

$$=$$

$$\{ \varphi_j \mid \Phi(j,n) \leq (f \circ t)(n) \text{ a.e.} \}.$$

Corollary:

$\Phi = dtime$. Then, $\forall$ total p.c. $f(n) \geq n$, $\exists$ total p.c. t s.t.

$$DTIME(t(n)) = DTIME(f(t(n))).$$

Moral: "$\exists$ arbitrarily large gaps in the DTIME complexity hierarchy."

Quiz: Does this violate the time-hierarchy theorem:

$$DTIME(o(t(n))) \neq DTIME(t(n) \log t(n))\ ?$$

Speedup Theorem [Blum, '71]:

$\forall\, \Phi$, $\forall$ total p.c. f, $\exists$ total p.c. g s.t. $\forall\, \varphi_i = g$,

$\exists\, \varphi_j = g$ satisfying:

$$f(n, \Phi(j, n)) < \Phi(i, n) \quad \text{a.e.}$$

Corollary:

$\Phi = dtime$. Then, $\forall$ total p.c. f, $\exists$ a language L s.t. $\forall$ DTMs M_i that decide L, $\exists M_j$ that decides L and that satisfies:

$$f(dtime(j,x)) < dtime(i,x) \quad \text{a.e.}$$

Moral: "$\exists$ computable functions with no asymptotically fastest algorithm because they can be endlessly sped up."

Union Theorem [McCreight-Meyer, '69]:

Let $f_1, f_2, f_3, \ldots$ be a computable list of total p.c. functions s.t. $\forall i, n$, $f_i(n) \leq f_{i+1}(n)$. Then, $\forall \Phi$, $\exists$ total p.c. f s.t.

$$\mathcal{C}_f^{\Phi} = \bigcup_i \mathcal{C}_{f_i}^{\Phi}.$$

Corollary/Moral:

$\exists$ total p.c. f_1, f_2, and f_3 s.t.:

- $P = DTIME(f_1)$
- $NP = NTIME(f_2)$
- $PSPACE = SPACE(f_3)$

Thank You!

Slides

Gap Theorem ($\Phi = dtime$):

$\forall$ total p.c. $f(n) \geq n$, $\exists$ total p.c. t s.t.

$$DTIME(t(n)) = DTIME(f(t(n))).$$

Proof:

- Enumerate DTMs $M_1, M_2, M_3, \dots,$ with $T_i(x) = dtime(i, x)$.
- $\forall n$, define

$$t(n) := \min\{m \mid \forall (i \le n)\,(T_i(n) \le f(m) \Rightarrow T_i(n) \le m)\}.$$

$f(m) \ge m$

- $t(n)$:

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m = 0
for i ∈ [n]:
    if m < T_i(n) ≤ f(m):
        m = T_i(n)
return m.
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- If M_i runs in time $f(t(n))$, then $T_i(n) \le t(n)$.

So,

$$\mathrm{DTIME}(t(n)) = \mathrm{DTIME}(f(t(n))).$$

QED

Speedup Theorem ($\Phi = dtime$):

$\forall$ total p.c. f, $\exists$ a language L s.t. $\forall$ DTMs M_i that decide L, $\exists\, M_j$ that decides L and satisfies:

$$f(dtime(j,x)) < dtime(i,x) \quad \text{a.e.}$$

Proof Idea:

- Let $f^n = f \circ f \circ \cdots \circ f$ n times.
- Construct (diagonalization) $L \subseteq \{0\}^*$ s.t.

  (i) $\forall M_i$ accepting L,

  $$\mathrm{dtime}(i, 0^n) > f^{n-i}(2) \text{ a.e.}$$

  (ii) $\forall k$, $\exists M_j$ accepting L s.t.

  $$\mathrm{dtime}(j, 0^n) \leq f^{n-k}(2) \text{ a.e.}$$

- Then,

  (ii) $\Rightarrow \exists M_j$ accepting L s.t. $\mathrm{dtime}(j, 0^n) \leq f^{n-i-1}(2)$ a.e.

  So

  (i) $\Rightarrow f(\mathrm{dtime}(j, 0^n)) \leq f^{n-i}(2) < \mathrm{dtime}(i, 0^n)$.

Union Theorem:

Let $f_1, f_2, f_3, \ldots$ be a computable list of total p.c. functions s.t. $\forall i, n$, $f_i(n) \leq f_{i+1}(n)$. Then, $\forall \Phi$, $\exists$ total p.c. f s.t.

$$C_f^{\Phi} = \bigcup_i C_{f_i}^{\Phi},$$

where

$$C_f^{\Phi} := \{ \varphi_i \mid \Phi(i,n) \leq f(n) \text{ a.e.} \}.$$

Proof Idea:

- Construct f (diagonalization) satisfying:

  (i) $\forall k \quad f(n) \geq f_k(n)$ a.e.

  (ii) $\forall i$, if

  $$(\forall k : \Phi(i,n) > f_k(n) \text{ i.o.}),$$

  then

  $$f(n) < \Phi(i,n) \text{ i.o.}$$

  "infinitely often"

- Then,

  (i) $\Rightarrow \quad C^{\Phi}_{f_j} \subseteq C^{\Phi}_{f}$

  (ii) $\Rightarrow$ If $\Phi(i,n) \leq f(n)$ a.e., then $\exists f_j$ s.t.

  $$\Phi(i,n) \leq f_j(n).$$

  $$\Rightarrow \qquad C^{\Phi}_{f} \subseteq \bigcup_i C^{\Phi}_{f_i}.$$