Quantum Complexity as Geometry

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Part I : Geometry

- Manifolds and Lie groups
- Parallel transport and geodesics

Part II : Quantum Complexity

- Postulates of Quantum Mechanics
- Quantum Circuits
- BQP

Part III : Quantum Complexity as Geometry

↳ (arXiv: quant-ph/0502070
arXiv: quant-ph/0603161)

Part I: Geometry

Def: A space M is a **manifold** iff $\forall p \in M$,

$\exists$ open set U containing p and a map

$$\phi : U \longrightarrow \phi(U) \subseteq \mathbb{R}^n$$

s.t.

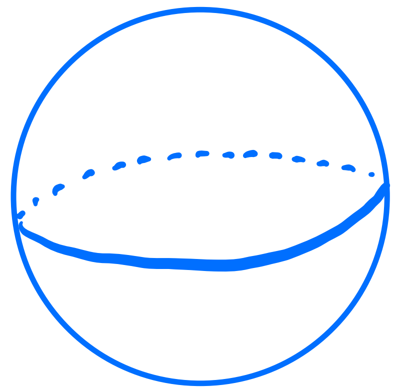
(i) ϕ is invertible
(ii) ϕ is continuous
(iii) ϕ^{-1} is continuous

} "ϕ is a homeomorphism"

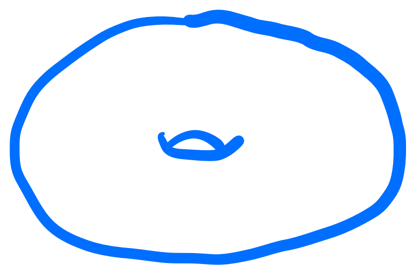
Intuition: M "looks like" $\mathbb{R}^n$, at least locally.

Ex:

- $M = S^2$



- $M = T^2$



Ex:

- $M = \{A \in \mathbb{R}^{n \times n} \mid A \text{ invertible}\}$
$=: GL(n, \mathbb{R})$

"the General Linear group of degree n"

Claim:

$GL(n, \mathbb{R})$ is <u>both</u> a group and a manifold!

That is, $GL(n, \mathbb{R})$ is a <u>Lie group</u>.

Proof $M = GL(n,\mathbb{R})$ is a manifold:

- $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$

  $\Longrightarrow$ $\mathbb{R}^{n\times n}$ a manifold

- $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ is continuous

- $GL(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$

  $\Longrightarrow$ $GL(n,\mathbb{R})$ is open subset of $\mathbb{R}^{n\times n}$

- Open subset of manifold is itself a manifold. $\blacksquare$

Ex:

- $M = \{ \mathcal{U} \in \mathbb{C}^{n \times n} \mid \mathcal{U}^\dagger \mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = I \}$

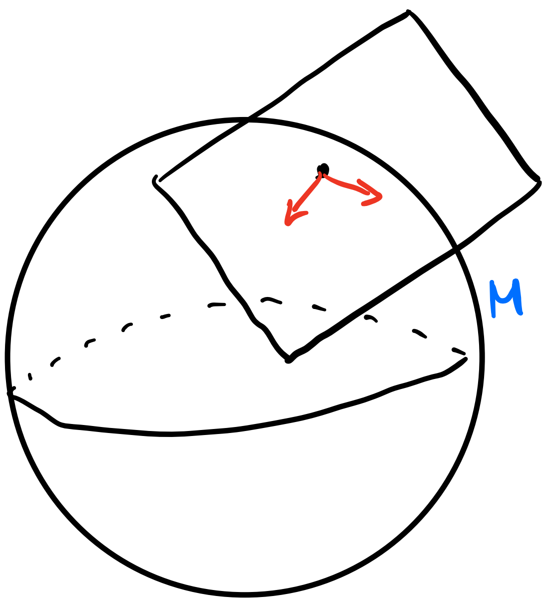
  $=: U(n)$

"the unitary group of degree n.

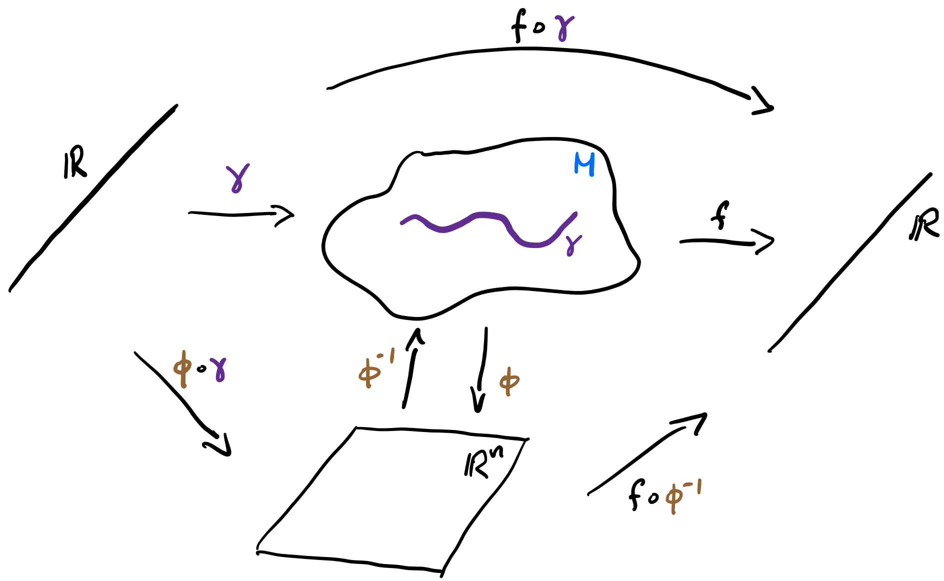
Claim: $U(n)$ is a Lie group.

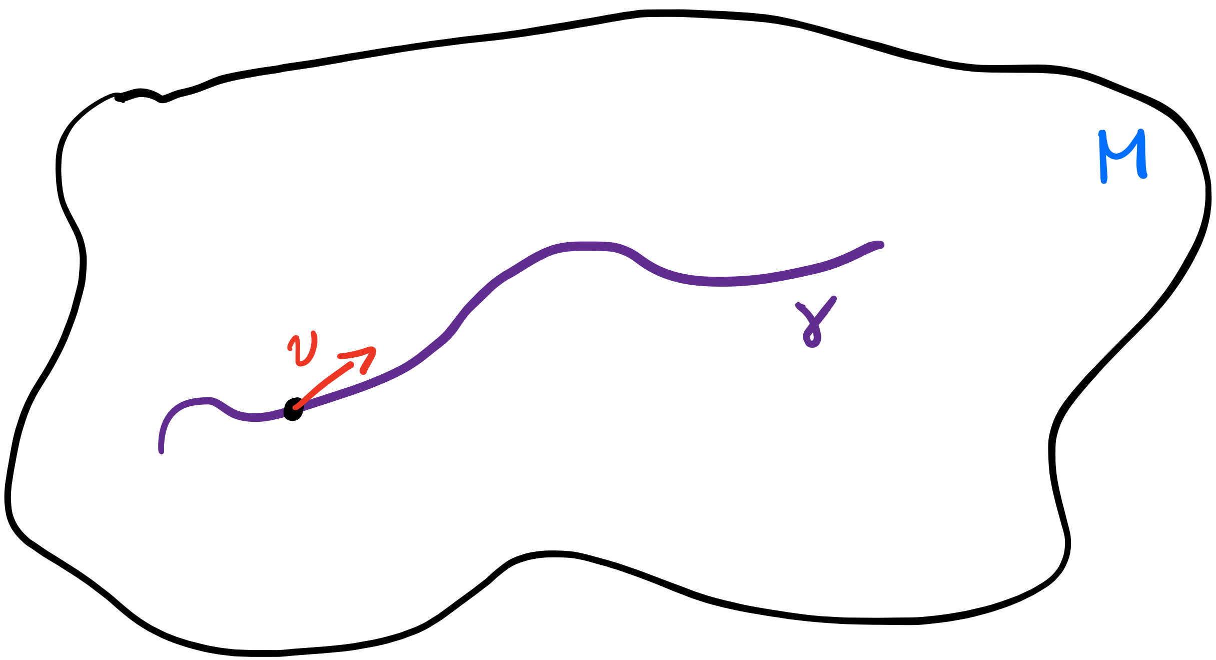
Vectors on Manifolds

Explicit Approach



Implicit Approach



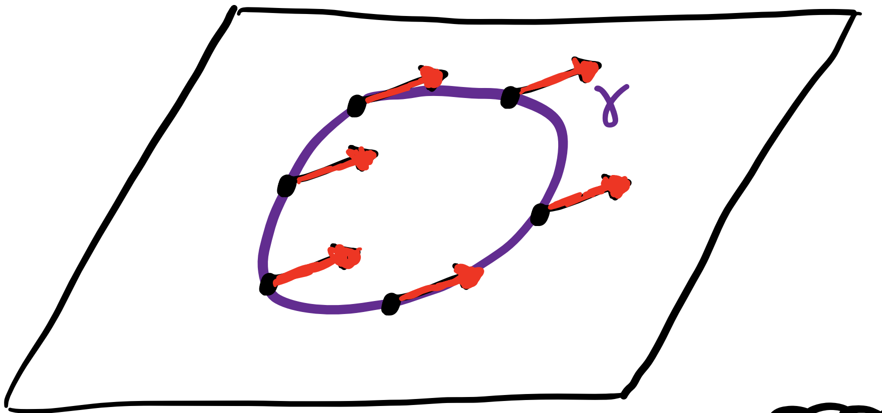


<u>Q</u>: How does $v = \sum_i v^i e_i$ change along γ?

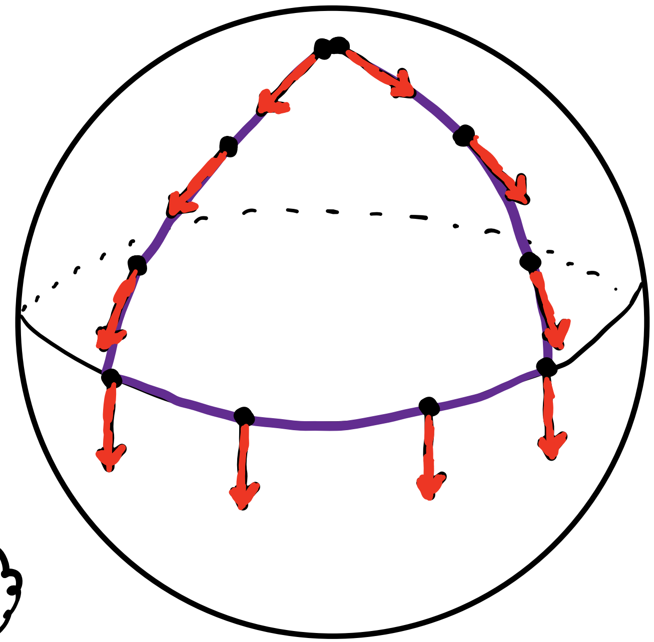
$$\left.\frac{dv}{dt}\right|_\gamma = \left.\frac{d\left(\sum_i v^i e_i\right)}{dt}\right|_\gamma = \sum_i \left.\left(\frac{dv^i}{dt} e_i + v^i \frac{de_i}{dt}\right)\right|_\gamma$$

Parallel Transport, i.e., Proof the Earth is not Flat

$\mathbb{R}^2$



S^2

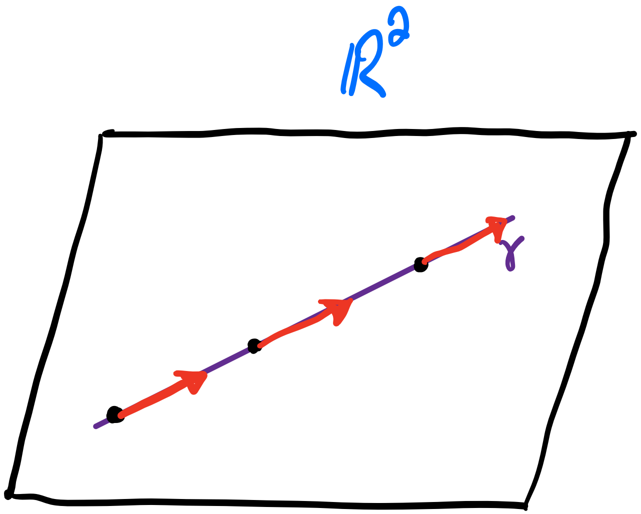


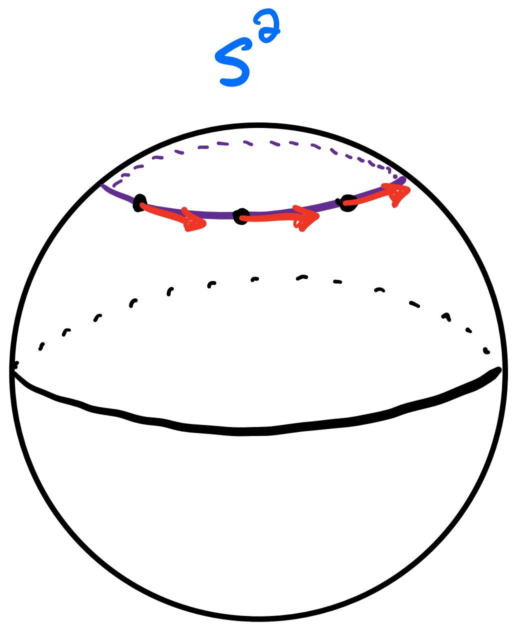
"connection coefficients"

$$\left.\frac{dv}{dt}\right|_\gamma = 0 \iff \frac{dv^i}{dt} + \sum_{b,c} \Gamma^i_{bc} \frac{dx^b}{dt} v^c = 0$$

<u>upshot:</u> M curved iff $\Gamma^a_{bc} \neq 0$ in all coordinate systems.

Def: A geodesic is a curve γ in M along which the tangent vector to γ is parallel transported.





Thm: If γ is a geodesic with coordinates $x^i = \phi^i \circ \gamma$, then

$$\frac{d^2 x^i}{dt^2} + \sum_{b,c} \Gamma^i_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt} = 0 .$$

Def: The geodesic equation is:

$$\frac{d^2 x^i}{dt^2} + \sum_{b,c} \Gamma^i_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt} = 0 .$$

Note: This is n 2^{nd} order ODEs, so specifying $x^i_{initial}$ and $\frac{dx^i_{initial}}{dt}$ completely determines the geodesic.

Def: A Riemannian manifold M is a manifold equipped with a Riemannian metric g, i.e., a (positive-definite) inner product g_p for all $p \in M$.

Def: Given $\gamma : [0,1] \to M$, the <u>length</u> of γ is

$$L_g[\gamma] := \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt .$$

Claim: Geodesics on a Riemannian manifold have extremal length.

Part II: Quantum Complexity

Some Postulates of Quantum Mechanics

Let S be a "quantum system".

(i) State of S $\longleftrightarrow$ $|\psi_S\rangle \in \mathbb{C}^n$

Ex: S = spin of an electron, $|\psi_S\rangle \in \mathbb{C}^2$.

- $|\psi_S\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ "spin-down"
- $|\psi_S\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ "spin-up"
- $|\psi_S\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ "spin superposition"

Some Postulates of Quantum Mechanics

(ii) Probability S is in state $|x_S\rangle$ $\longleftrightarrow$ $|\langle x_S | \psi_S \rangle|^2$

(iii) State evolution of S from $|\psi_S\rangle$ to $|\phi_S\rangle$ $\longleftrightarrow$ $|\phi_S\rangle = \mathcal{U}|\psi_S\rangle$, $\mathcal{U} \in U(n$

Upshot: A new type of randomized computer!

Quantum Computation in a Very Primitive Nutshell

S = Quantum system

$A = \{|x_{S,1}\rangle, |x_{S,2}\rangle, \dots, |x_{S,\ell}\rangle\}$ (set of "accept" states)

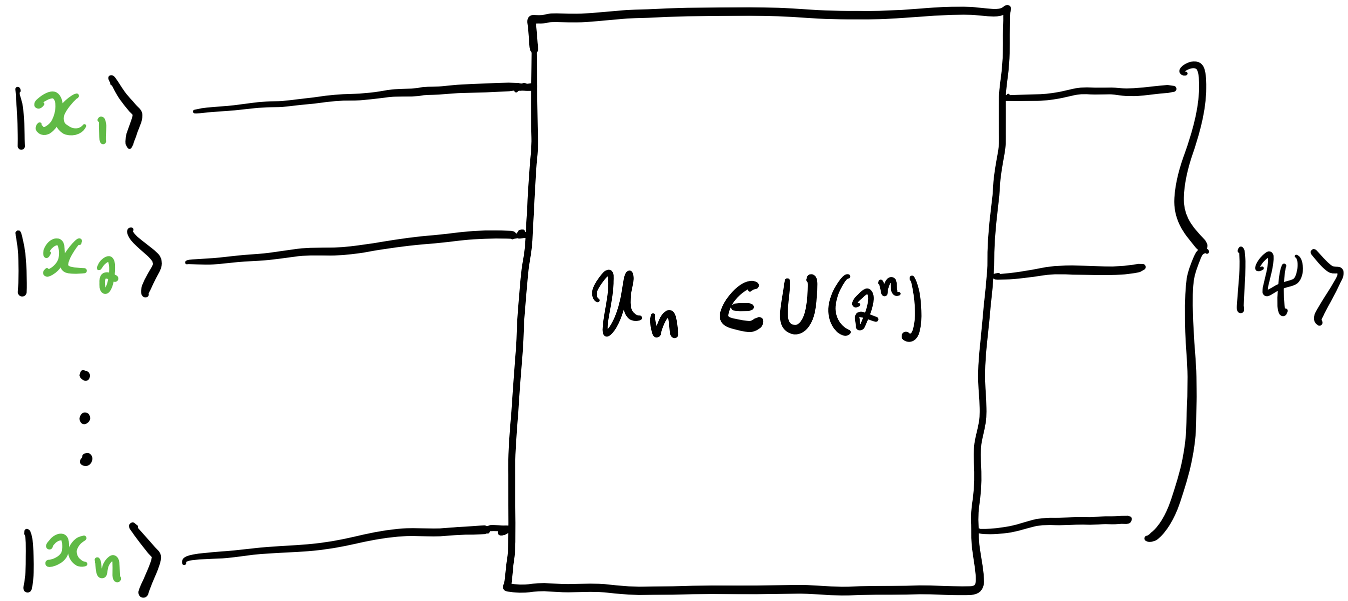
- First, evolve:

$$|\psi_S\rangle \xrightarrow{\;\mathcal{U} \in U(n)\;} |\phi_S\rangle = \mathcal{U}|\psi_S\rangle$$

- Then, "accept" iff:

$$\Pr\left[|\phi_S\rangle \in A\right] = \sum_{i=1}^{\ell} \left|\langle x_{S,i}|\phi_S\rangle\right|^2 \geq 2/3$$

Let $L \subseteq \{0,1\}^*$. Given $x \in \{0,1\}^*$, do



Then, "accept" x iff:

$$\Pr[\text{first qubit of } |\psi\rangle \text{ is } |L(x)\rangle] \geq 2/3 .$$

Caveat: Need

$$n \mapsto U_n$$

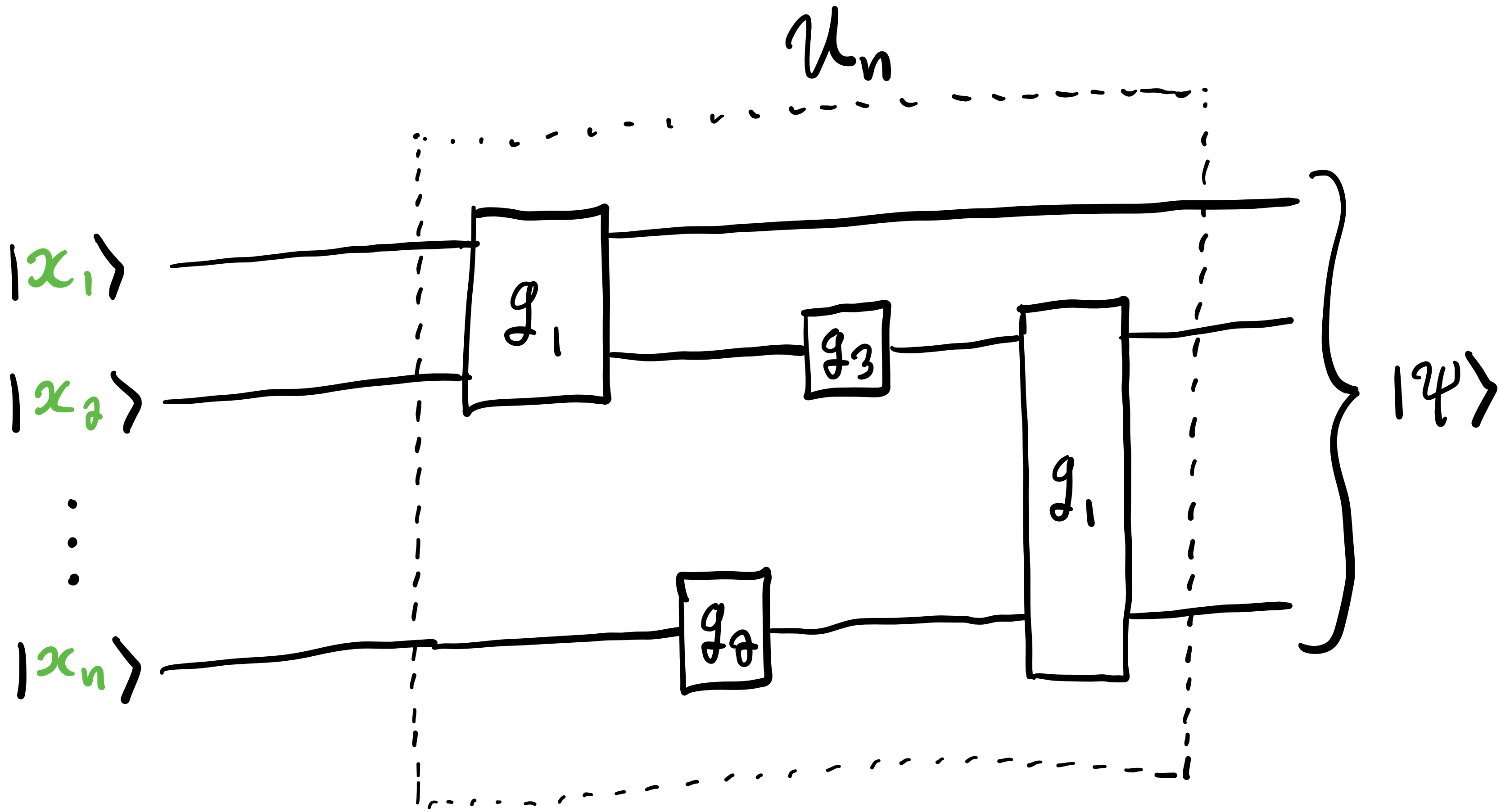
to be efficiently computable.

Fact: Like how $\{\wedge, \vee, \neg\}$ is universal for Boolean circuits, $\exists$ universal quantum gate sets G.

Solution: Can write each U_n as product of gates in G:

$$U_n \sim g_\ell \, g_{\ell-1} \cdots g_1$$

If $\ell = \mathrm{poly}(n)$, then <u>efficient</u>.



U_n
$|x_1\rangle$
$|x_2\rangle$
$|x_n\rangle$
g_1
g_3
g_2
g_1
$|\psi\rangle$

Def: Let G be a universal gate set.

$$L \in BQP$$

$$\Longleftrightarrow$$

$\exists$ uniform, poly-size family $(U_n)_{n \in \mathbb{N}}$ of quantum circuits over G s.t. $\forall x \in \{0,1\}^*$,

$$\Pr\left[\text{first qubit of } U_{|x|}|x\rangle \text{ is } |L(x)\rangle\right] \geq 2/3 .$$

Q: $BPP \stackrel{?}{=} BQP$

Idea: Toward understanding BQP, important to understand optimally-sized quantum circuit families $\mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}}$.

Def: Given universal G and a family $(\mathcal{U}_n)_{n \in \mathbb{N}}$, define

$$M_G(\mathcal{U}_n) := \min \{ \ell \mid \mathcal{U}_n \sim g_\ell \cdots g_1, \, g_i \in G \}.$$

Part III: Quantum Complexity as Geometry

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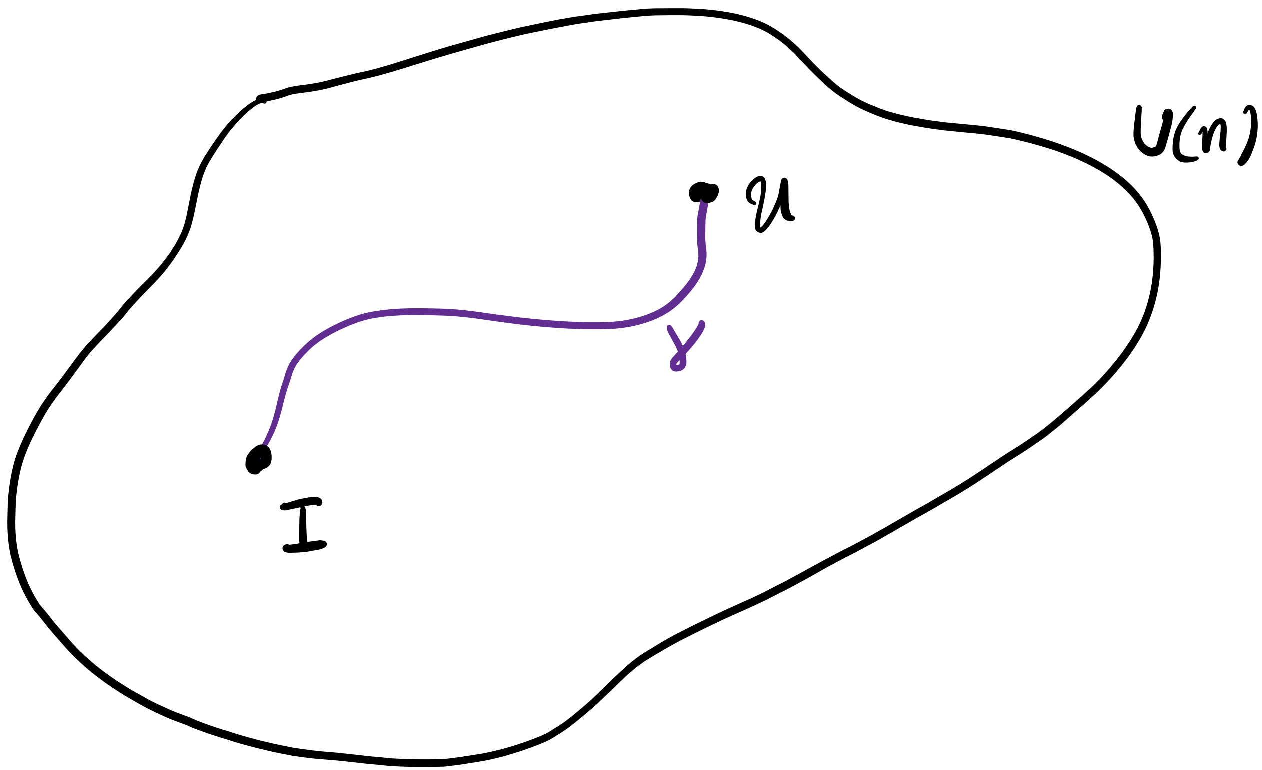
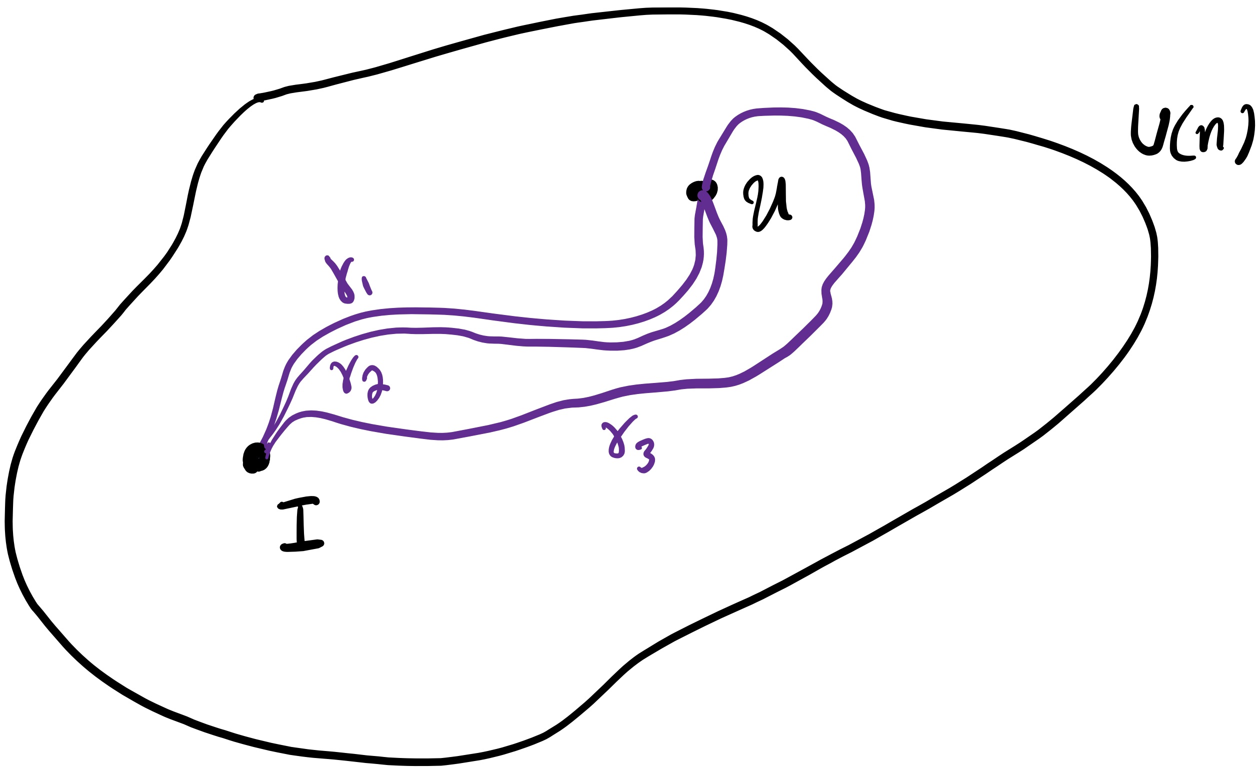
<u>Recall:</u> $U(n)$ is a Lie group (and hence a manifold).



Def. Let F be a Riemannian metric on $U(n)$. Define

$$d_F(I, \mathcal{U}) := \min_{\substack{\gamma :\, \gamma \text{ connects} \\ I \text{ and } \mathcal{U}}} L_F[\gamma].$$



Thm. (Nielsen et al.):

$\exists$ a metric F on $U(2^n)$ s.t.

(i) $d_F(I, \mathcal{U}_n) \leq m_G(\mathcal{U}_n)$

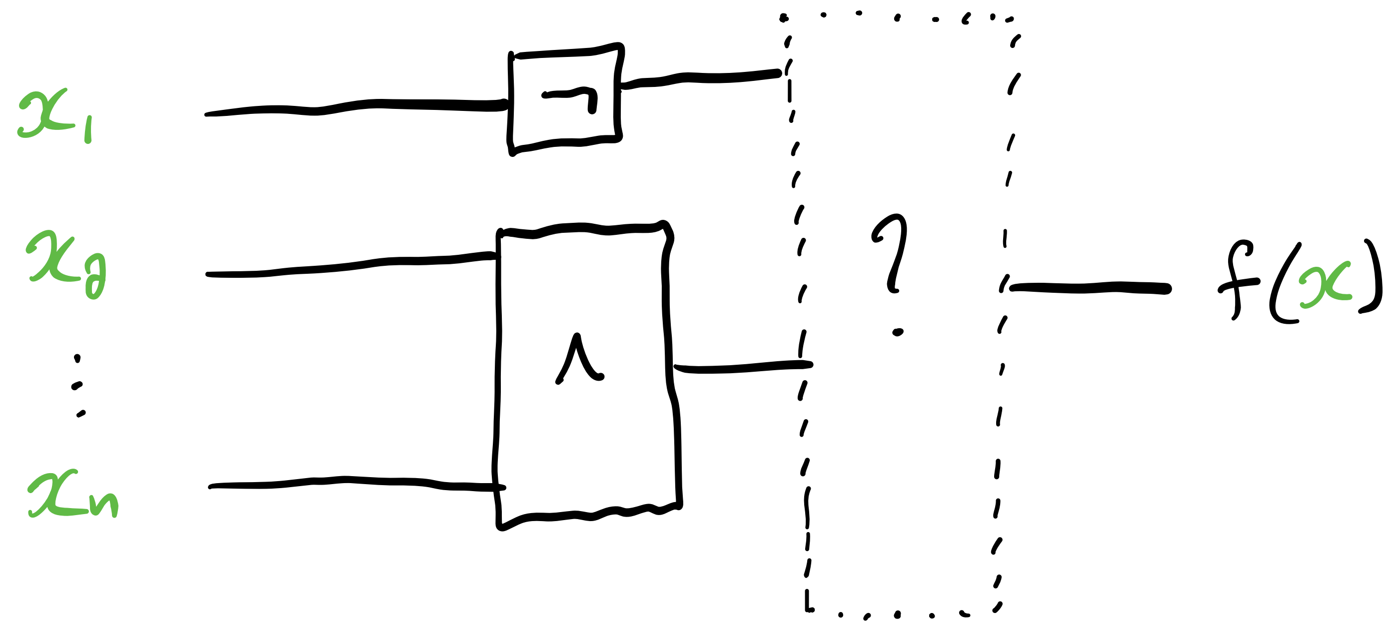
(ii) $d_F(I, \mathcal{U}_n) = \text{poly}(n) \Rightarrow m_G(\mathcal{U}_n) = \text{poly}(n)$.

That is,

(i) Geodesics lower bound circuit complexity

(ii) Poly-size geodesics imply poly-size circuit complexity.

Q: Let $C : \{0,1\}^n \to \{0,1\}$ be optimal Boolean circuit that computes $f : \{0,1\}^n \to \{0,1\}$. Given only part of C, how do you construct the rest of C?



Corollary (Very Roughly):

As a geodesic, which obeys

$$\frac{d^2x^i}{dt^2} + \sum_{b,c} \Gamma^i_{bc} \frac{dx^b}{dt}\frac{dx^c}{dt} = 0,$$

is completely determined by x^i_{initial} and $\frac{dx^i_{\text{initial}}}{dt}$, it is possible to determine the optimal quantum circuit from just its initial data.

Thank You!

- Schrödinger equation

$$\frac{dU}{dt} = -iH(t)\,U(t)$$

- Impose "cost" $F(H(t))$ on Hamiltonian H s.t. F defines Riemannian geometry on $U(2^n)$ (similar to VQE's, minus the geometry).